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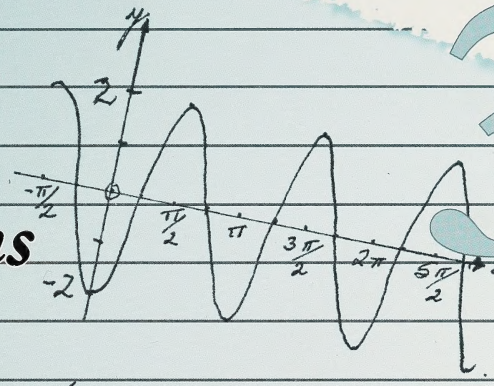
Module 2

MATHEMATICS 30

Exponents and Logarithms



$y = 2\sin 2(x - \pi/4)$

When $x = \pi/2$,

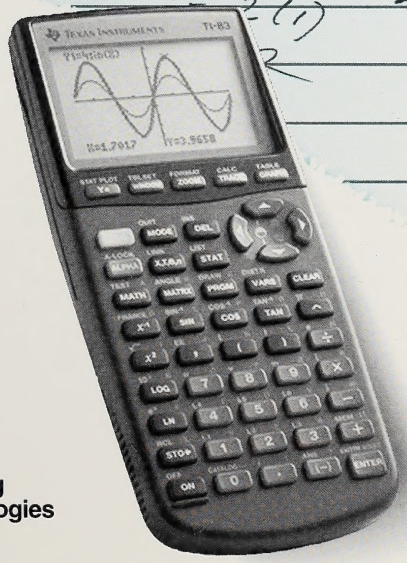
$y = 2\sin 2(\pi/2 - \pi/4)$

$= 2\sin 2(\pi/4)$

$= 2\sin \pi/2$

$= 2(1)$

$= 2$



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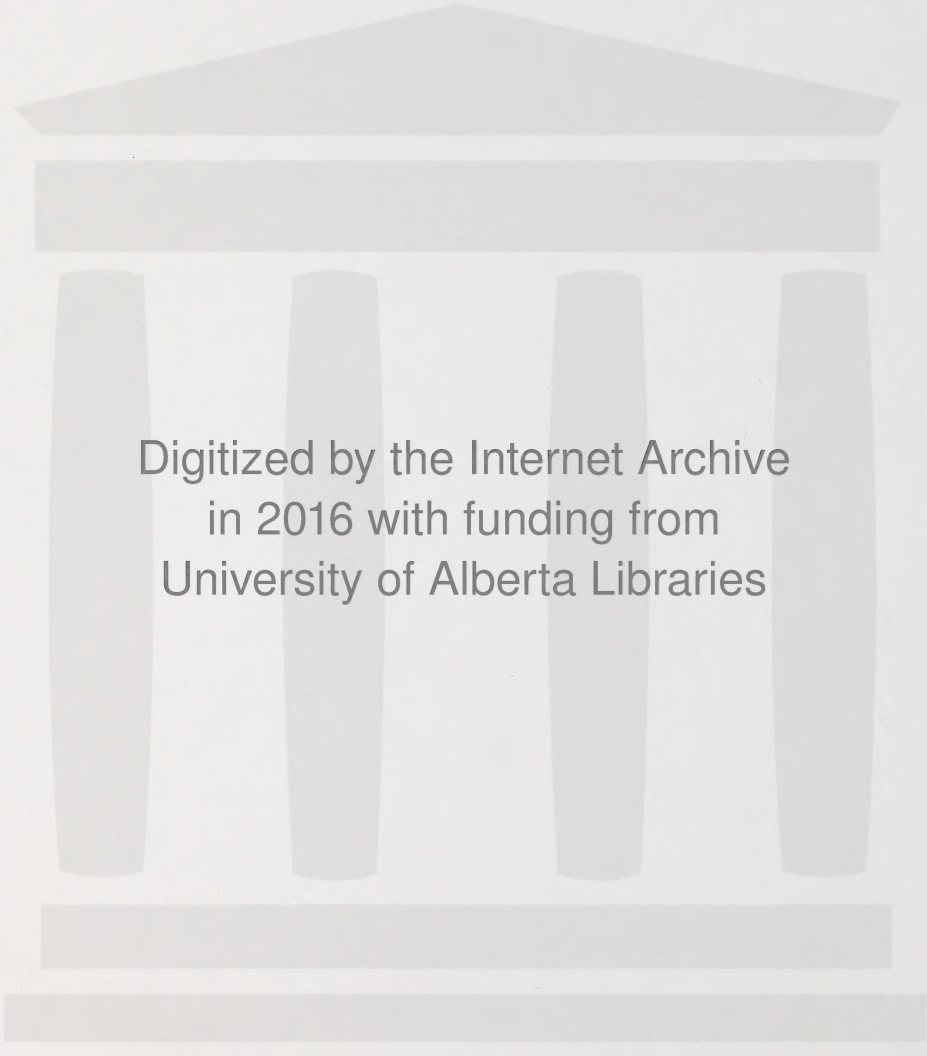
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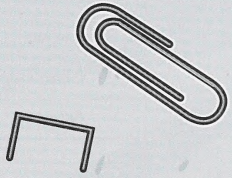
Pure MATHEMATICS 30

Module 2

Exponents and Logarithms



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Pure Mathematics 30
Module 2: Exponents and Logarithms
Student Module Booklet
Learning Technologies Branch
ISBN 0-7741-1642-0

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Parents	
General Public	
Other	

The Learning Technologies Branch has an Internet site that you may find useful. The address is as follows:

http://www.learning.gov.ab.ca/ltb

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Welcome

Pure Mathematics 30

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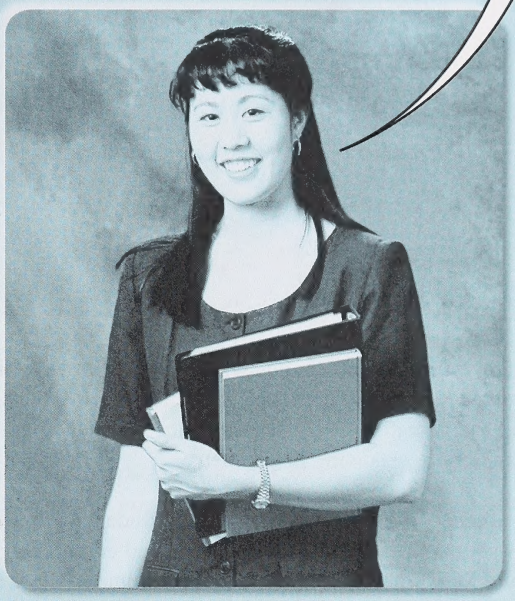
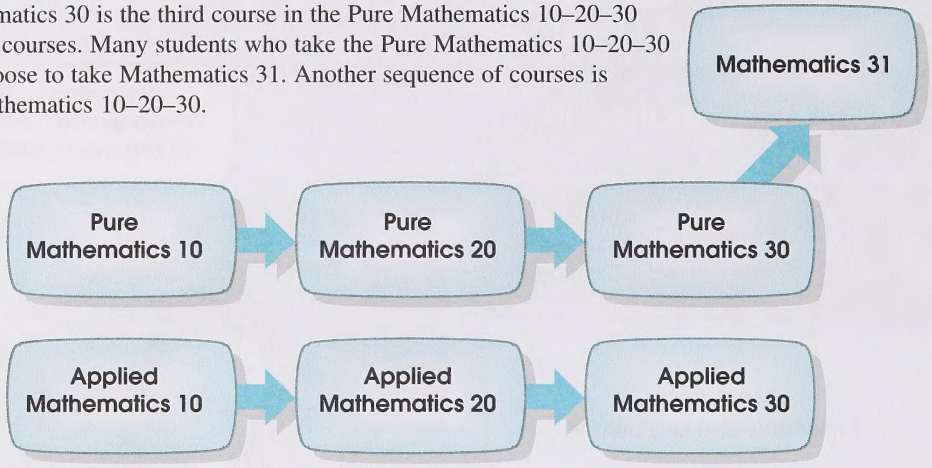
  Probability and Statistics



Pure Mathematics 30 contains seven modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.

High School Mathematics Courses

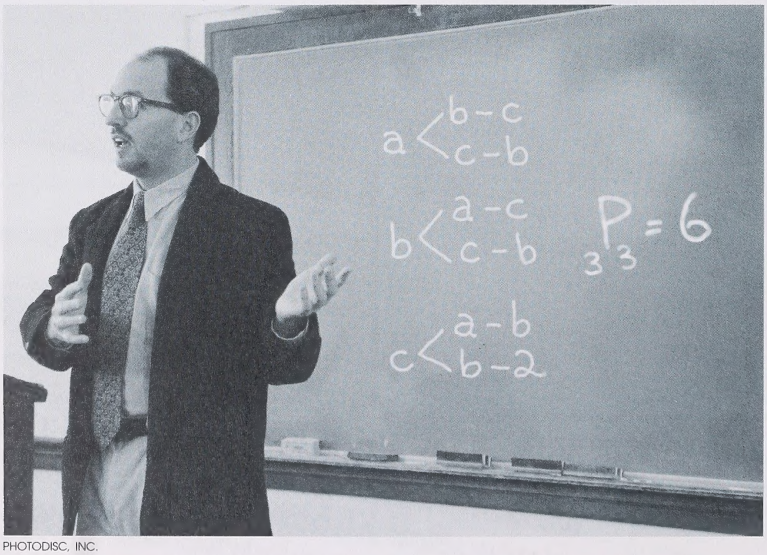
Pure Mathematics 30 is the third course in the Pure Mathematics 10–20–30 sequence of courses. Many students who take the Pure Mathematics 10–20–30 will also choose to take Mathematics 31. Another sequence of courses is Applied Mathematics 10–20–30.



How Do the Sequences Differ?

Each sequence of courses is designed for students with different mathematical strengths and interests. The Pure Mathematics 10–20–30 sequence is intended for students who are strong in mathematical theory and algebra. The Applied Mathematics 10–20–30 sequence is better suited to students who prefer to solve mathematical problems using numerical reasoning or geometry.

Each sequence of courses is designed for students with different career plans. Students who plan to take courses such as commerce, engineering, physics, and engineering technologies in post-secondary education will need to take Pure Mathematics 30 in high school. Some of these students will also need to take Mathematics 31.



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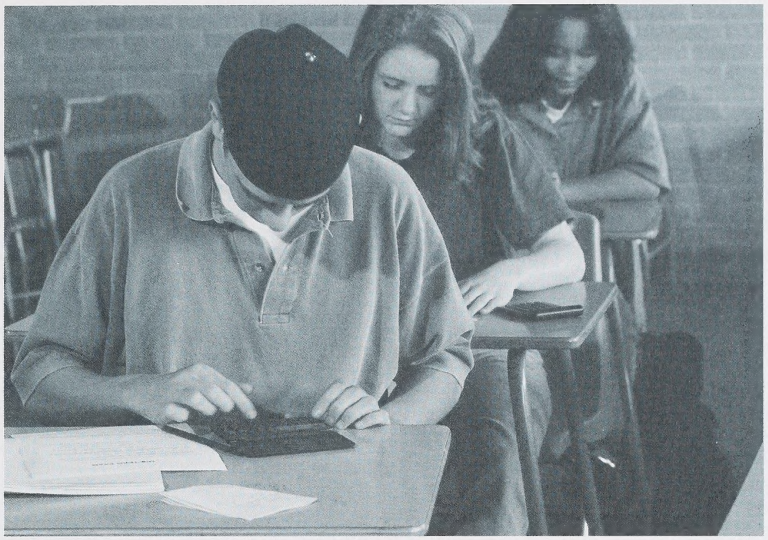
Where Can I Obtain Diploma Examination Information?



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Alberta Students will write a diploma examination at the end of the course. Alberta Learning provides several documents to help students prepare for this examination. These documents are found in the directory under "Students and Learning" on the Alberta Learning website, **http://www.learning.gov.ab.ca**. Information like course expectations, the makeup of the diploma examination, keyed copies of previous examinations, preparation guides, and calculator policies are available to students at this site.

Each year, in February and September, Alberta Learning provides teachers with information on a **student project**, which teachers **may** use as part of your overall assessment. Information to students will also be posted on the Alberta Learning website. Check with your teacher to determine what you will be expected to do. Be aware that one of the diploma examination's written-response questions is worth 10% of your diploma examination mark and will deal with elements of this project.

You should take advantage of the many sources of information about Pure Mathematics 30. Your success depends on your understanding of course expectations and evaluation procedures. Work closely with your teacher and do not hesitate to ask questions.

Remember, take the initiative to find out all you can about Pure Mathematics 30.

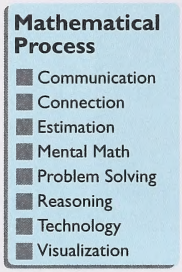
Can I Transfer to the Applied Mathematics Sequence of Courses?

If you do not successfully complete Pure Mathematics 30, you can repeat the course or you can transfer to the Applied Mathematics sequence. If you decide to transfer to the Applied Mathematics sequence, you may take the five-credit course called Applied Mathematics 20b or you may take the five-credit course called Applied Mathematics 20.

Mathematical Processes

Throughout the course you will be expected to do the following:

- communicate mathematically
- connect mathematical ideas to everyday experiences and to concepts in other disciplines
- develop and use estimation strategies
- develop and use mental math strategies
- develop and use problem-solving strategies
- reason and justify your thinking
- select and use appropriate technologies as tools to solve problems
- use visualization to assist in processing information, making connections, and solving problems



This icon will show which mathematical process you are expected to perform.

In order to develop these mathematical processes more fully, you are encouraged to ask someone who is also taking Pure Mathematics 30 to be your study partner. You may find that having a friend to discuss mathematics with makes your studying more enjoyable.

You are also encouraged to record your mathematical ideas in a journal. Here are some suggestions for organizing your journal:

- Keep a section on new concepts and procedures you have learned. Get in the habit of describing new concepts, procedures, and strategies in your own words. If you are having difficulty, write your questions in your journal before discussing them with your teacher or study partner. Record useful ways to help you remember what a concept means. Make graphic organizers (such as context webs, Venn diagrams, or organizational charts) to help you connect mathematical ideas.

- Keep a section on your mathematical accomplishments. This can include solutions to problems that you are proud of solving. It can also include landmark events, such as when you finally grasped a difficult concept (an "aha!") or when you first used a calculator or spreadsheet program in a new way.

- Keep a section on mathematics in the everyday world. This could include newspaper articles, cartoons, and descriptions of careers that require a mathematical background.

Resources You Will Need

In order to complete Pure Mathematics 30 successfully, you will need to have access to the following resources, which can be purchased through the Learning Resources Distributing Centre:

- the *MATHPOWER™ 12* textbook, Western Edition, published by McGraw-Hill Ryerson (2000)
- a graphing calculator

  Any of the following graphing calculators may be used: *Texas Instruments* TI-83, TI-83 Plus, TI-82, TI-86, TI-89, TI-92, or TI-92 Plus; *Casio* CFX-9850Ga-Plus, CFX-9850G, CFX-9800G, or FX-9700 series; or *Sharp* EL-9600c, EL-9600, EL-9300, or EL-9200. **Note:** Some of these calculators are no longer commercially manufactured.
- a computer and a spreadsheet program

  Any of the following spreadsheet programs may be used: *ClarisWorks™ 4.0* (or higher) for Macintosh, *ClarisWorks™ 4.0* (or higher) for Windows, *Microsoft® Works 3.0* (or higher) for Windows, or *Microsoft® Access 97* and *Microsoft® Excel 97* for Windows (a part of *Microsoft® Office Professional 95* or *Microsoft® Office Professional 97*).

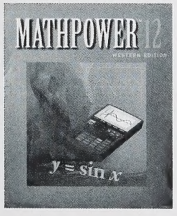
Note: Examples in this course show the TI-83 graphing calculator and the *Microsoft® Excel 97* spreadsheet program where applicable. If you have access to a videocassette recorder, you may wish to view the video *The TI-83 Graphing Calculator Video Tutor* to review some of the calculator's features. Check with your distance learning provider concerning the availability of this video.

Finally, you will need the following resources, which can be purchased locally:

- a mathematical instrument set (compass, protractor, ruler, and triangles)
- a notebook or binder in which to respond to the questions asked
- a notebook or binder for journal writing

Other Visual Cues

In addition to the Mathematical Process icon described earlier, you will find other visual cues throughout this course. Read the explanations given to discover what each icon prompts you to do.



Refer to the textbook.

View a videocassette.

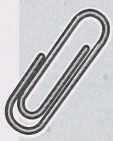
Use the Internet to explore a topic.

Answer the questions in the Assignment Booklet.

Use the Pure Mathematics 30 Companion CD.

Remember: *Any Internet website address given in this module is subject to change.*

CONTENTS



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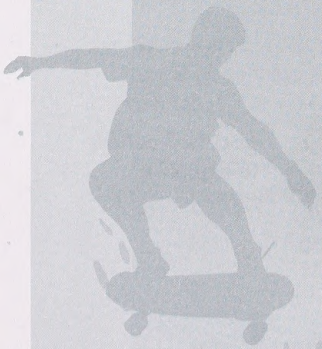

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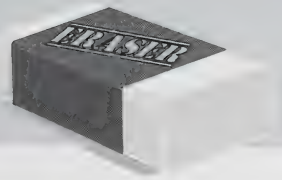


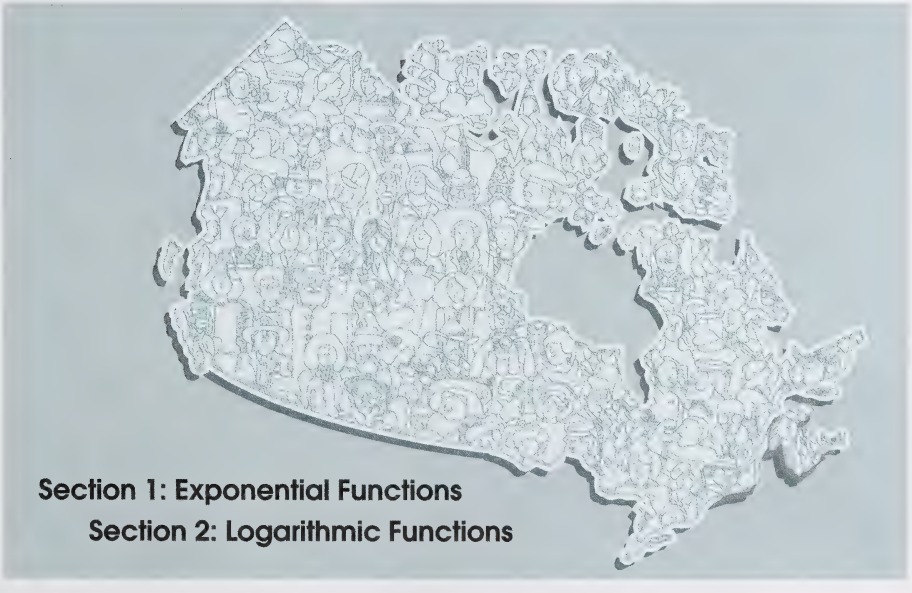
Module Overview

At the beginning of the twenty-first century, Canada's population was more than 30 million. This represents about a 5.5-fold increase over the last 100 years, or an annual increase of approximately 1.72%. Do you think the population will continue to increase at this rate? What do you think the nation's population will be at the end of the twenty-first century? Assuming the growth rate does not change, what would the population be after 1000 years?

Population growth is often modelled by a special class of functions, called exponential functions. In Section 1, you will explore the exponential function and its graph using technology. You will then use exponential functions and equations arising from the application of exponential functions to solve a variety of practical problems—problems that include population growth, compound interest, depreciation, and radioactive decay.

In Section 2, you will investigate another class of functions, called logarithmic functions. You will discover that logarithmic and exponential functions are related and that many laws for logarithms are similar to the laws for exponents. You will apply these laws and properties to solve equations and model practical problems in growth and decay similar to those introduced in Section 1.

Module 2: Exponents and Logarithms



Evaluation

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Exponents and Logarithms, has two sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

In this module, you will be directed to the accompanying Assignment Booklets. You are expected to complete two Assignment Booklets for this module. Your grading in this module is based upon the assignments you submit for evaluation. The mark distribution is as follows:

Assignment Booklet 2A	
Section 1 Assignment	**35 marks**
Assignment Booklet 2B	
Section 2 Assignment	**50 marks**
Final Module Assignment	**15 marks**
TOTAL	**100 marks**

Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklets, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 30.

Good luck!

Section 1

JOE MCDONALD/CORBIS

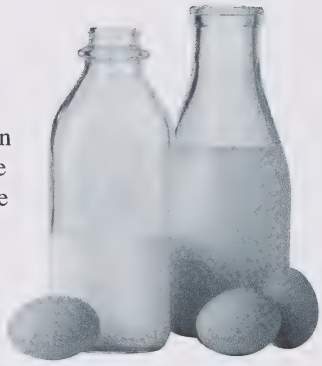
Exponential Functions

Nature, when in balance, keeps insect populations under control. If unchecked, insect numbers would increase very quickly. The growth of insect populations, under ideal conditions, is often modelled by a special class of functions, called exponential functions.

Exponential functions are useful in describing a variety of situations involving growth or decay. For example, they can be used to predict the growth of human populations, the increase in the number of bacteria in a culture, the amount of an investment bearing compound interest, the spread of epidemics, and the decay of radioactive isotopes.

In this section, you will explore exponential functions. You will investigate their graphs, analyse them with the aid of technology, find the roots of exponential equations, and model and solve real-world problems like growth and decay.

Activity 1: Growth and Decay



Canadians take the availability of inexpensive, safe, high-quality food for granted. Milk, for example, is a staple. Processed in modern dairies, it is pasteurized to eliminate disease-causing bacteria and reduce the number of spoilage micro-organisms. One method of pasteurization is to pass the milk through heated tubing so its temperature rises to 71°C and remains at 71°C for 15 s. Not all spoilage micro-organisms are eliminated, but a sufficient number are, thus greatly increasing the milk's storage life.

If you have access to the Internet, you can find out more about pasteurization and its inventor, Louis Pasteur, by searching for "Pasteur" and "pasteurization." Information about pasteurization can be found at the following website:

http://www.jaybeeprecision.com/applju~1.htm

One measure of how the numbers of a particular micro-organism can be reduced by heating to a particular temperature is the decimal reduction time, D. This is the time required to reduce the numbers by a factor of 10. For example, if there are 10 000 micro-organisms originally, there would be only 1000 micro-orgnisms remaining after time D, only 100 micro-organisms after time $2D$, and so on. Because there are 10 000 micro-organisms to begin with, the population, y, after time t is described by the function $y = 10\,000(0.1)^{\frac{t}{D}}$. This function is an example of an exponential function, a function of the form $y = Ab^x$, where A is a non-zero constant and $b > 0$ and $b \neq 1$.

In this activity, you will explore exponential functions, describe their graphs, and use them to model growth or decay.

Turn to page 77 of *MATHPOWER™ 12* and read "Exponential Functions as Models of Growth or Decay."

1. Answer the following questions on pages 77 and 78 of the textbook:

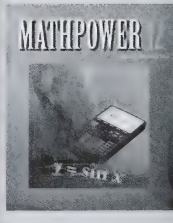
a. "Explore: Develop a Model"
b. questions 1 to 7 of "Inquire"

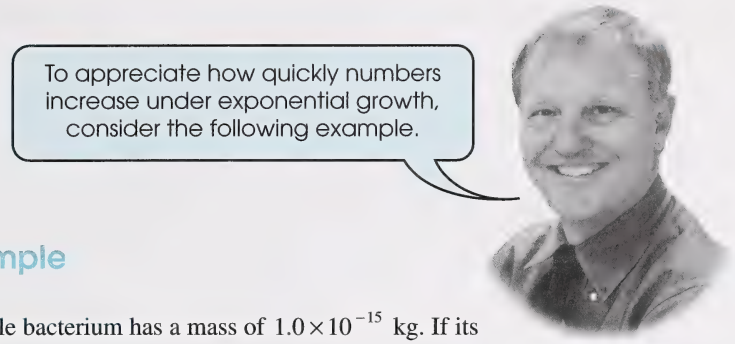
Turn to page 63 to compare your responses with those in the Appendix.

Mathematical Process

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

a function of the form $y = Ab^x$





Example

A single bacterium has a mass of 1.0×10^{-15} kg. If its doubling time is 30 min and if there are sufficient nutrients and the wastes they produced do not inhibit growth, what would the total mass of the bacteria be after 3 days? How would that mass compare to the mass of Earth, which is approximately 6.0×10^{24} kg?

Solution

Let y be the number of bacteria at any time, t (in hours). Since the initial population is 1 bacterium and the number is doubling every 30 min (or 0.5 h), the function that models the population after t hours is

$$y=1\times2^{\frac{t}{0.5}}$$

Convert days to hours.

$$3\text{d}\times24\text{ h/d}=72\text{ h}$$

Now, find the number of bacteria after 3 days.

$$\begin{aligned}y&=1\times2^{\frac{72}{0.5}}\\&\doteq2.23\times10^{43}\end{aligned}$$

Multiply the number of bacteria by the mass of 1 bacterium.

$$\begin{aligned}\text{Mass}&\doteq2.23\times10^{43}\times1.0\times10^{-15}\text{ kg}\\&\doteq2.23\times10^{28}\text{ kg}\end{aligned}$$

The total mass of the bacteria would be about 2.23×10^{28} kg.

Compare the mass of the bacteria with the mass of Earth.

$$\frac{m_{\text{bacteria}}}{m_{\text{Earth}}} \doteq \frac{2.23 \times 10^{28}}{6.0 \times 10^{24}}$$
$$\doteq 3716.790\,866$$

The mass of the bacteria would be approximately 3700 times the mass of Earth.

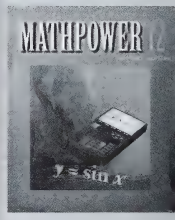
growth that can be modelled by an exponential function

the time it takes for a population to double

doubling time

In the preceding example, you investigated **exponential growth** in terms of its **doubling time**. For bacteria, the time required for the population to double is also called the **generation time**. However, other rates of increase may be given. In the next example, you will explore exponential growth of a bacteria and the function that models it, given the time it takes for the population to triple.

Turn to page 78 of *MATHPOWER™ 12* and work through Example 1.



2. Answer the following questions on page 82 of the textbook:

a. questions 16 and 18 to 23 of "Practice"
b. question 25 of "Applications and Problem Solving"

Turn to page 64 to compare your responses with those in the Appendix.

Example

a. Explain how the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 3(2)^x - 4$ can be obtained by transforming the graph of $y = 2^x$.

b. On the same axes, draw the graphs of the three functions in part a. Check using your graphing calculator.

c. State the domain, range, intercepts, and asymptotes of each graph.

Solution

a. Rewrite $y=\left(\frac{1}{2}\right)^x$.

$$\begin{aligned} y &= \left(\frac{1}{2}\right)^x \\ &= \left(2^{-1}\right)^x \\ &= 2^{-x} \end{aligned}$$

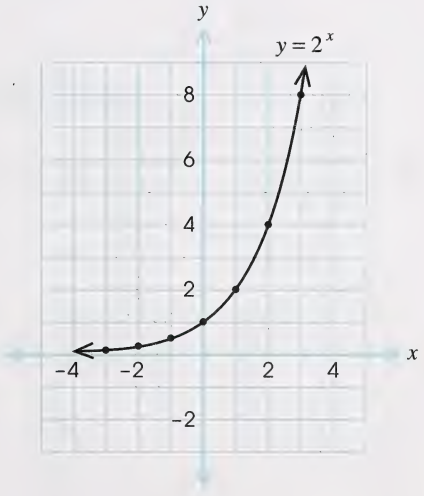
Recall from Module 1 that $y=f(-x)$ is the mirror image of $y=f(x)$ in the y-axis. Therefore, to obtain the graph of $y=\left(\frac{1}{2}\right)^x$, reflect the graph of $y=2^x$ in the y-axis.

To obtain the graph of $y=3(2)^x-4$ from the graph of $y=2^x$, stretch the graph of $y=2^x$ vertically by a factor of 3 and then translate the result 4 units down.

b. First, draw the graph of $y=2^x$ from a table of values.

$y=2^x$

x	-3	-2	-1	0	1	2	3
y	0.125	0.25	0.5	1	2	4	8



Use the proceeding table of values and the transformations described in part a. to obtain tables for the other two functions; then plot the resulting points.

Because $y=\left(\frac{1}{2}\right)^x$ is the reflection of $y=2^x$ in the y-axis, simply change the signs of the x-values in the table of values of $y=2^x$.

$\boldsymbol{y=\left(\frac{1}{2}\right)^x}$

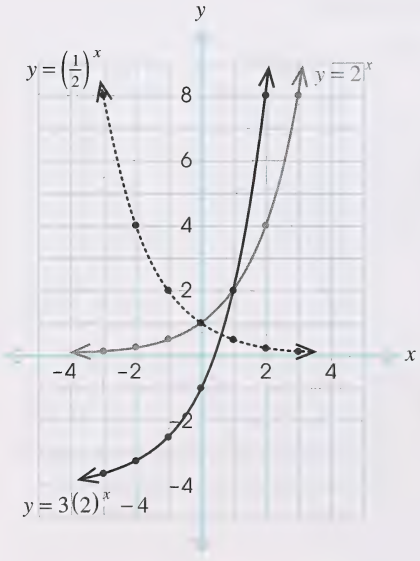
x	-3	-2	-1	0	1	2	3
y	8	4	2	1	0.5	0.25	0.125

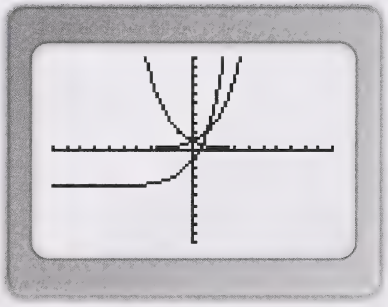
To obtain a table of values for $y=3(2)^x-4$, multiply the y-values of $y=2^x$ by 3 and subtract 4. This will stretch the graph of $y=2^x$ vertically by a factor of 3 and then translate the result 4 units down.

eg., $y=3(0.125)-4$
$=0.375-4$
$=-3.625$

$\boldsymbol{y=3(2)^x-4}$

x	-3	-2	-1	0	1	2	3
y	-3.625	-3.25	-2.5	-1	2	8	20





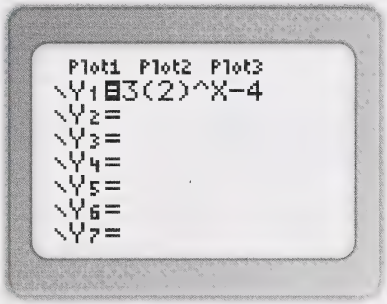
c.

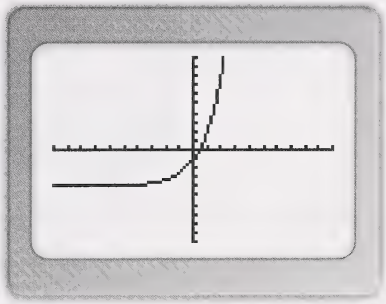
Graph	$y=2^x$	$y=\left(\frac{1}{2}\right)^x$	$y=(3)2^x-4$
Domain	Reals	Reals	Reals
Range	$y\geq 0$	$y\geq 0$	$y\geq -4$
x-intercept	None	None	0.415 037 5
y-intercept	1	1	-1
Asymptote	$y=0$	$y=0$	$y=-4$

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

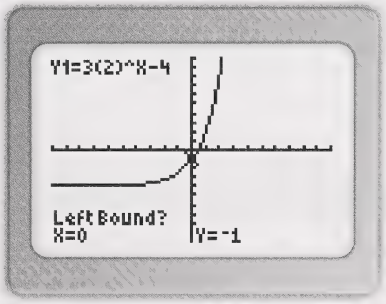
To find the *x*-intercept of $y=3(2)^x-4$ on your graphing calculator, use the Zero feature from the CALCULATE menu. Begin by entering the function.

Y= 3 (2) ^ X,T,Θ,*n* − 4 GRAPH

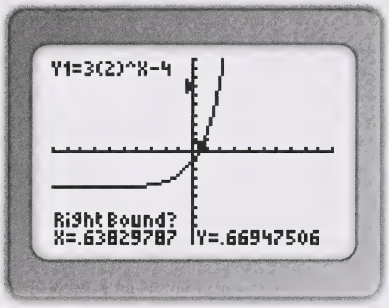




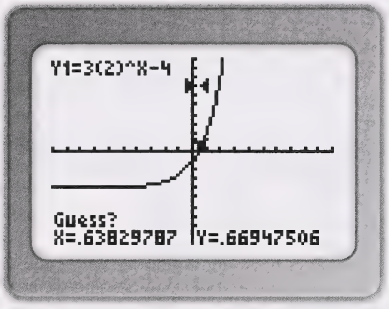
Next, access the Zero feature by pressing 2nd [CALC] 2.



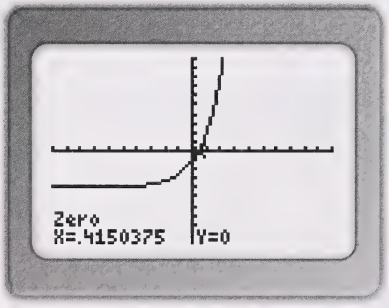
Notice that the cursor is located on the y-axis. Because this point is close to the x-intercept, you can use it as the left bound. Press ENTER.



Now, the calculator prompts you to choose the right bound. Use the arrow keys to position the cursor above the x-axis; then press ENTER.



The calculator now prompts you to enter your guess. Tap the left arrow key once, positioning the cursor between the left and right bounds, and press ENTER.



Now, answer the following questions regarding the graphs of exponential functions.

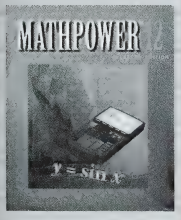
3. a. For $y = Ab^x$, where $b > 0$ and $b \neq 1$, state the domain, range, intercepts, and asymptotes, if any. Sketch a diagram to illustrate your answers.

b. Explain why the restrictions $b > 0$ and $b \neq 1$ are placed on the exponential function.

Turn to page 69 to compare your responses with those in the Appendix.

Did you know that the formula for compound interest, which you studied in Pure Mathematics 10 and Pure Mathematics 20, is an example of an exponential function?

The formula for compound interest is $A(t) = P(1+i)^n$, where $A(t)$ is the accumulated amount, P is the principal, i is the interest rate per compounding period, and n is the number of periods. In the next question set, you will explore compound interest and other applications of exponential functions. You will use the graphs of exponential functions to assist you.



Turn to pages 79 to 81 of *MATHPOWER™ 12* and work through Examples 2 and 3.

4. Answer the following questions on pages 81 to 83 of the textbook:

a. questions 1, 5, 13, 14, and 15 of "Practice"
b. questions 24, 26, 27, 28, and 31 of "Applications and Problem Solving"

Turn to page 70 to compare your responses with those in the Appendix.

For convenience, the accumulated amount of a deposit earning compound interest is shown as a steadily increasing, continuous curve when graphed over time. However, you should be aware that interest is only added at the end of each interest period and that the amount on a deposit is constant between interest periods.

Example

Anna deposits $100 in an account bearing interest at 10%, compounded annually. The interest is deposited into her account at the end of each year.

a. Graph the amount of money she has in her account over a 5-year period. Also, graph the exponential function you used to calculate the accumulated amount on your graphing calculator.

b. Use your graphs to determine when her original investment will grow by 50%.

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Solution

a. Use the compound interest formula, $A(t) = P(1+i)^n$, to determine the amount in the account at the end of each year.

$P = 100$, $i = 10\% = 0.10$, and $n = t$

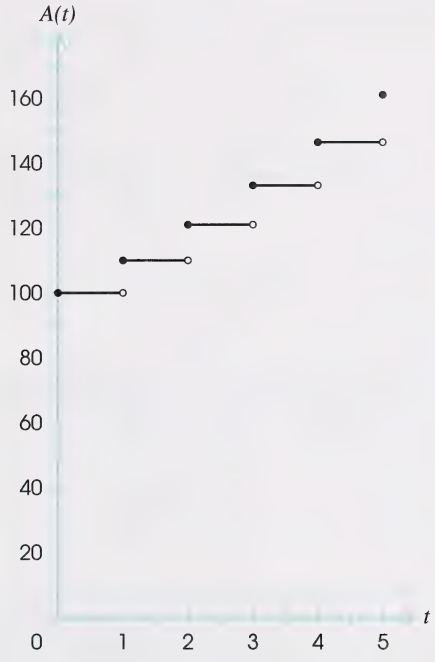
$$\therefore A(t) = 100(1+0.10)^t$$
$$= 100(1.1)^t$$

An exponential function for the amount of the investment after t years is $A(t) = 100(1.1)^t$.

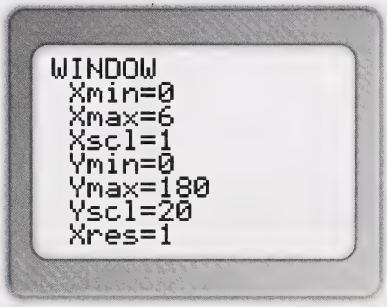
Time	Amount ($)
Present	100
End of 1st year	$100(1.1)^1 = 110.00$
End of 2nd year	$100(1.1)^2 = 121.00$
End of 3rd year	$100(1.1)^3 = 133.10$
End of 4th year	$100(1.1)^4 = 146.41$
End of 5th year	$100(1.1)^5 = 161.05$

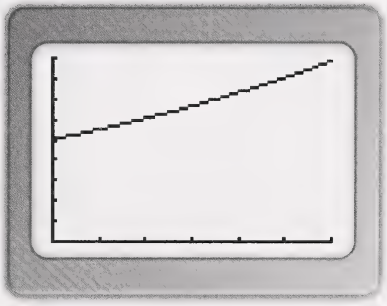
Graph these values. **Remember:** The amount is constant until the interest is added. That means the graph is discontinuous. This is an example of a step function—a function whose graph increases in discrete amounts and is constant between increments.

a function whose graph increases in discrete amounts and is constant between increments



Now, graph the function $A(t) = 100(1.1)^t$ on your graphing calculator using the window settings shown.





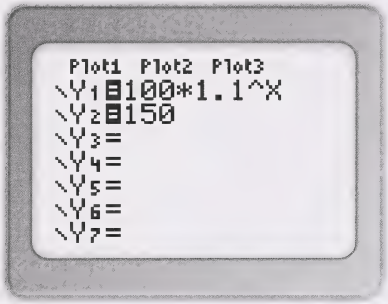
When you see the graph, you should remember that it is an approximation of the actual situation.

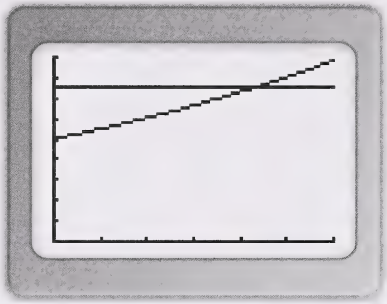
b. When Anna's investment grows by 50%, she will have

$$\$100 + 50\% \text{ of } \$100 = \$100 + \$50$$
$$= \$150$$

If you look at the first graph, Anna never has exactly \$150.00. She will have to wait 5 years for her investment to grow by at least 50%. At the end of the fifth year, she will have \$161.05.

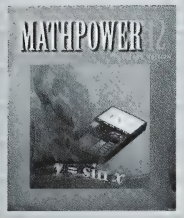
To find this answer using your graphing calculator, graph $y = 150$ as Y_2 and determine where the two graphs intersect.





Use the Intersect feature from the CALCULATE menu to determine where the two graphs intersect.

Recall that interest is added at the end of each year; so, even though 4.254 163 7 is closer to 4 years, Anna must wait 5 years before her original investment will increase by at least 50%.



In the preceding example, you used your graphing calculator to graph the exponential function by entering its equation. Suppose you had a table of values, or a set of points, that you thought could be described by an exponential function. How can you use your graphing calculator to determine the equation of this function?

Turn to "TECHNOLOGY" on page 91 of *MATHPOWER™ 12* and read "Exponential Regression."

5. Answer questions 1 to 7 of Investigation 1, "Are University Tuition Fees Rising Exponentially?" on pages 91 and 92 of the textbook.

6. Use your graphing calculator to determine the exponential function that passes through $(-2, 0.25)$, $(-1, 0.5)$, $(0, 1)$, $(1, 2)$, and $(2, 4)$.

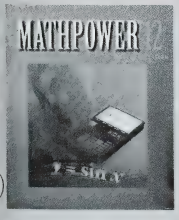
Turn to page 76 to compare your responses with those in the Appendix.

Caution: Your graphing calculator gives you the equation of the exponential function that best fits your data. This does not guarantee that the data is necessarily exponential. For example, the following table of values, obtained from $y = x^2$, yields $y = ab^x$, where $a = 0.248\,053\,255\,9$ and $b = 3.117\,958\,065$.

x	0.5	1	2	3	4
y	0.25	.1	4	9	16

Mathematical Process

- Communication
- Connection
- Estimation
- [x] Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization



Now Try This

A definite asset when working with exponential functions is the ability to calculate powers mentally. Skill in this area will help you detect computational errors in your solutions to problems involving exponents, thus assisting you in determining whether or not your answer is reasonable.

7. Turn to page 71 of *MATHPOWER™ 12* and answer questions 13 to 24 of "Mental Math: Working with Exponents."

Turn to page 81 to compare your responses with those in the Appendix.

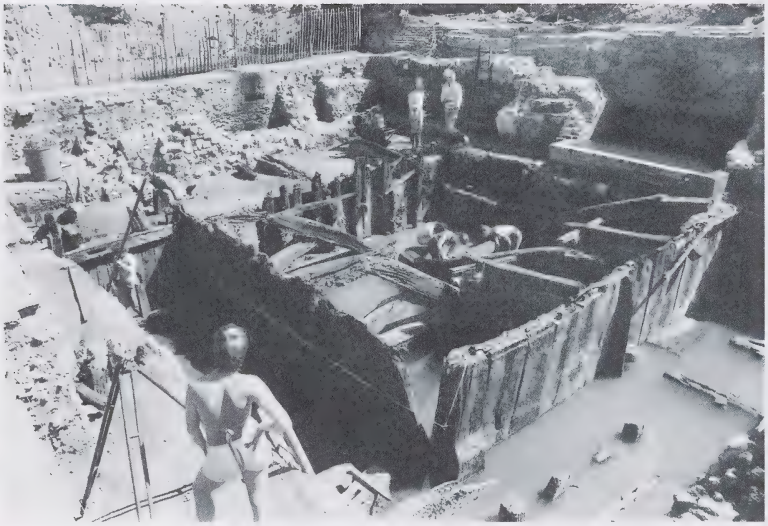
Looking Back

In this activity, you explored how exponential functions can be used to model real-world situations involving growth and decay.

In your journal, prepare a list of situations involving exponential growth and a second list involving exponential decay. Explain how the exponential functions that model these two categories differ. Use examples to illustrate your explanation.

Mathematical Process
- [x] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

Activity 2: Exponential Equations



HULTON-DEUTSCH COLLECTION/CORBIS

Are you interested in the ancient peoples and their cultures? Have you considered archaeology as a career? One of the tools archaeologists use to determine the age of an artifact made from plants or animals is carbon dating. This procedure is based on the principle that plants and animals, while alive, utilize the various isotopes of carbon as basic building blocks. Carbon-14, an unstable isotope of carbon continually being produced in the upper atmosphere by the bombardment of nitrogen atoms, is one of these isotopes. It decays at a constant rate. While an organism is alive, its relative proportion of carbon-14 compared to the more stable and abundant carbon-12 is constant, because new carbon-14 atoms are constantly replacing those that undergo radioactive decay back into nitrogen. When the plant or animal dies, the carbon-14 atoms that decay are not replaced. In the laboratory, when the artifacts are examined, the fraction of remaining carbon-14 is measured and compared to what it would have been when the plant or animal was alive.

Mathematical Process
- [ ] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

the time it takes for a radioactive substance to decay to one-half its original amount

an equation in which the variable occurs in an exponent

The **half-life** of carbon-14 is 5730 years. This is the time it takes for 50% of the original carbon-14 atoms to decay to nitrogen. The percentage, $P(t)$, of the original carbon-14 that remains after t years can be modelled by the following exponential function:

$$P(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

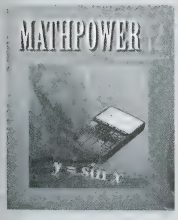
If the remaining percentage is 25%, then the age, t (in years), of the artifact can be found by solving the following **exponential equation**:

$$25 = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

An exponential equation is an equation obtained from an exponential function—the unknown appears in the exponent. In this activity, you will solve exponential equations and related problems.

Some of the problems you will encounter involve radioactive decay. If you have access to the Internet, you can find out more about carbon dating as an application of radioactive decay at the following website:

http://www.flmnh.ufl.edu/natsci/vertpaleo/aucilla10_1/Carbon.htm



Turn to page 86 of *MATHPOWER™ 12* and read "Solving Exponential Equations."

1. Answer the following questions on page 86 of the textbook:

a. "Explore: Use Trial and Error"
b. questions 1 to 6 of "Inquire"

Turn to page 81 to compare your responses with those in the Appendix.

One of the basic procedures for solving exponential equations is converting powers to equivalent powers with a common base. Consider the following example.

Example

Solve for x, and check your answer.

$$4^{x-2} = 8^{x-1}$$

Solution

Since $4 = 2^2$ and $8 = 2^3$, rewrite each power using base 2.

$$4^{x-2} = 8^{x-1}$$

$(2^2)^{x-2} = (2^3)^{x-1}$ ← Substitute 2^n for 4 and 2^n for 8.

$2^{2x-4} = 2^{3x-3}$ ← Simplify the exponents.

$2x - 4 = 3x - 3$ ← Equate the exponents.

$2x - 3x = -3 + 4$

$-x = 1$

$x = -1$

Note: If $b^x = b^y$, then $x = y$.

Check

LS	RS
4^{x-2}	8^{x-1}
$= 4^{-1-2}$	$= 8^{-1-1}$
$= 4^{-3}$	$= 8^{-2}$
$= \frac{1}{4^3}$	$= \frac{1}{8^2}$
$= \frac{1}{64}$	$= \frac{1}{64}$

LS = RS

Turn to page 87 of *MATHPOWER™ 12* and work through Example 1.

2. Answer the following questions on pages 89 and 90 of the textbook:

a. questions 1, 10, 13, and 16 of "Practice"
b. questions 24.a., 24.f., and 33 of "Applications and Problem Solving"

Turn to page 83 to compare your responses with those in the Appendix.

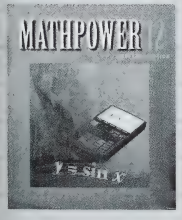
In Activity 1, you used exponential functions as models of growth and decay. In problem situations, these functions often lead to exponential equations. In the next part of this activity, you will use exponential equations to solve growth and decay problems.

Turn to page 87 of *MATHPOWER™ 12* and read from the red line to the red line near the bottom of page 89, working through Examples 2 to 4.

3. Answer the following questions on page 90 of the textbook:

a. questions 17, 18, 20, and 23 of "Practice"
b. questions 25 to 28 of "Applications and Problem Solving"

Turn to page 85 to compare your responses with those in the Appendix.



Now Try This

In many problem-solving situations, the ability to guess a reasonable answer and then check to see if it is right is a necessary part of your problem-solving toolkit. Not only does this approach help you narrow possibilities, it may also kindle insights into algebraic approaches to the problem.

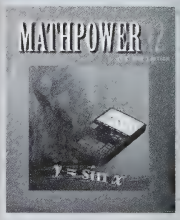
Turn to "PROBLEM SOLVING" on page 84 of *MATHPOWER™ 12* and read "Guess and Check."

4. Answer question 3 of "Applications and Problem Solving" on page 85 of the textbook. Answer this question by first using guess and check and then an algebraic approach.

Turn to page 93 to compare your response with the one in the Appendix.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☒ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization



Looking Back

In this activity, you examined exponential equations and how to convert powers to a common base to solve these equations.

Can all exponential equations be solved by using a common base? Why or why not? Record your answer in your journal, using examples to verify your reasoning. You may wish to discuss this question with another student taking Pure Mathematics 30.

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Students often have difficulty forming the correct exponential functions in problem situations involving growth or decay. You can avoid most mistakes by thinking of exponential functions modelling growth or decay as simple extensions of the compound interest formula. If you use the compound interest formula, $A(t) = P(1+i)^n$, as your starting point, you should be able to form the correct function every time.

Example

Form the exponential function that models each problem situation.

a. The population of a town of 10 000 residents in Saskatchewan is predicted to grow at an annual rate of 6%. What is the population, $f(t)$, after t years?

b. The intensity of light below the surface of a particular water hazard at a local golf course is reduced by 30% for every metre below the surface. What is the intensity, $I(d)$, of the light at depth d if the intensity at the surface is I_0?

c. The doubling period of a bacterium is 30 minutes. If there are 200 bacteria to begin with, state the number of bacteria, $N(t)$, after t minutes.

d. The half-life of strontium-90 is 25 years. Determine the amount, y, of strontium-90 remaining after t years if there were originally 2 g of strontium-90.

Solution

a. Work from $A(t) = P(1+i)^n$.

In this example, the population, $f(t)$, after t years corresponds to the amount, $A(t)$.

The current population, 10 000, corresponds to the original principal, P.

The annual rate of population increase, 6%, corresponds to the interest rate per period, i. Therefore, $i = 6\%$ or 0.06.

The increase in population is quoted as an annual rate. Therefore $n = t$.

Substitute this information into the interest formula to form the required exponential function.

$$A(t) = P(1+i)^n$$
$$f(t) = 10\,000(1+0.06)^t$$
$$= 10\,000(1.06)^t$$

Therefore, the exponential function that models the population growth is $f(t) = 10\,000(1.06)^t$.

b. Work from $A(t) = P(1+i)^n$.

The intensity of light, $I(d)$, at depth d corresponds to the amount, $A(t)$.

The intensity of light at the surface, I_0, corresponds to the original principal, P.

The decrease in intensity per metre increase corresponds to the interest rate per period. Therefore, $i = -30\%$ or -0.30.

> ***Remember:***
> *A decrease is negative.*

The decrease in intensity is expressed per metre of depth. Therefore, $n = d$.

Substitute this information into the interest formula to form the required exponential function.

$$A(t) = P(1+i)^n$$
$$I(d) = I_0\,(1-0.30)^d$$
$$= I_0\,(0.70)^d$$

Therefore, the exponential function that models the intensity of light is $I(d) = I_0\,(0.70)^d$.

c. Work from $A(t) = P(1+i)^n$.

The bacteria population, $N(t)$, after t minutes corresponds to the amount, $A(t)$.

The current bacteria population, 200, corresponds to the original principal, P.

If a population is doubled, then it increased by 100%. This increase corresponds to the interest rate per period, i. Therefore, $i = 100\%$ or 1.00.

The bacteria population doubles every 30 minutes. The number of doubling periods in t minutes is $\frac{t}{30}$. Therefore, $n = \frac{t}{30}$.

Substitute this information into the interest formula to form the required exponential function.

$$A(t) = P(1+i)^n$$
$$N(t) = 200(1+1.00)^{\frac{t}{30}}$$
$$= 200(2)^{\frac{t}{30}}$$

The exponential function that models the doubling time of a bacterium is $N(t) = 200(2)^{\frac{t}{30}}$.

d. Work from $A(t) = P(1+i)^n$.

The amount of strontium-90, y, remaining after t years corresponds to the amount, $A(t)$.

The current sample size, 2 g, corresponds to the original principal, P.

After one half-life, the sample size has decreased by 50%. This decrease corresponds to the interest rate per period, i. Therefore, $i = -50\%$ or -0.50.

The half-life is 25 years. The number of half-lives in t years is $\frac{t}{25}$. Therefore, $n = \frac{t}{25}$.

Substitute this information into the interest formula to form the required exponential function.

$$A(t) = P(1+i)^n$$
$$y = 2(1-0.50)^{\frac{t}{25}}$$
$$= 2(0.50)^{\frac{t}{25}} \text{ or } 2\left(\frac{1}{2}\right)^{\frac{t}{25}}$$

The exponential function that models the half-life of 25 of strontium-90 is $y = 2\left(\frac{1}{2}\right)^{\frac{t}{25}}$ or $y = 2(0.5)^{\frac{t}{25}}$.

Practise the technique shown in the preceding example in the next question set.

1. Turn to page 128 of *MATHPOWER™ 12* and answer questions 2.a. and 3.a. of "Chapter Check."

2. The value of a popular rookie hockey card is increasing at an annual rate of 20%. It is currently valued at \$150. Write an equation that gives the value, $V(t)$, of the card after t years.

Turn to page 94 to compare your responses with those in the Appendix.

Enrichment

Often, you cannot use a common base to rewrite the powers in an exponential equation. One approach is to use your graphing calculator to solve these equations.

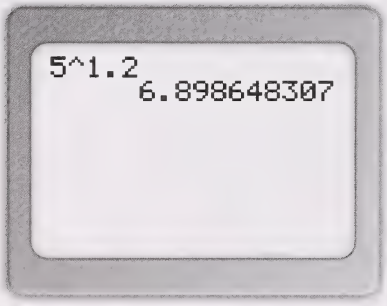
Example

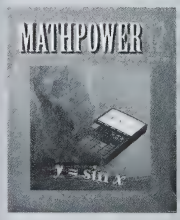
Solve $5^x = 8$.

Solution

Because $5^1 < 8 < 5^2$, $1 < x < 2$. You could use trial and error to obtain an approximation for x. Suppose you guessed $x = 1.2$.

5 ^ 1 • 2 ENTER



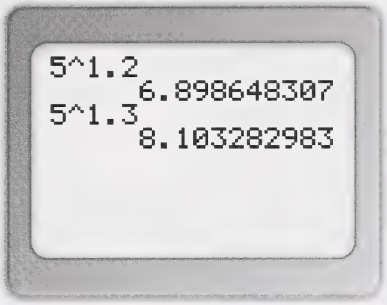


Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

The value of $x = 1.2$ appears too small. Try $x = 1.3$.

5 ^ 1 • 3 ENTER



Since $5^{1.3} \doteq 8.103\,282\,983$, $x \doteq 1.3$.

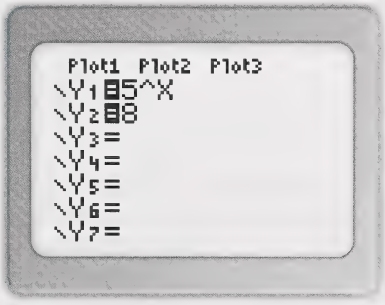
How could you obtain a more precise approximation in the preceding example? Simply think of the equation as arising from the solution to the following system of equations:

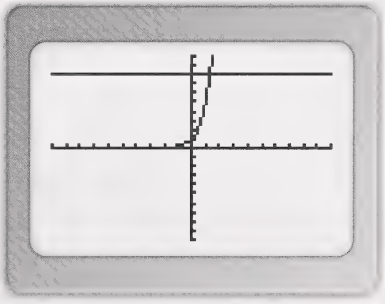
$y = 5^x$ (1)

$y = 8$ (2)

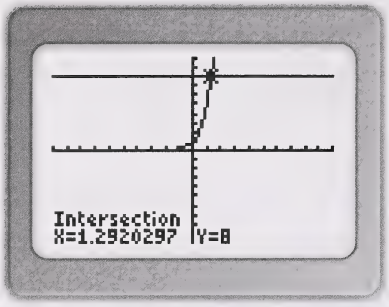
Graph these equations and determine where their graphs intersect. (Use the standard window settings.)

Y= 5 ^ X,T,Θ,*n* ENTER 8 GRAPH





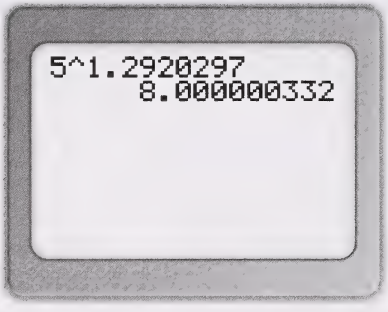
Next, use the Intersect feature from the CALCULATE menu to determine where the two graphs cross.



The x-value of the intersection point is the solution to the original exponential equation.

Check

Substitute $x = 1.292\,029\,7$ into (1).



Use the Intersect feature on your graphing calculator to solve the following equations.

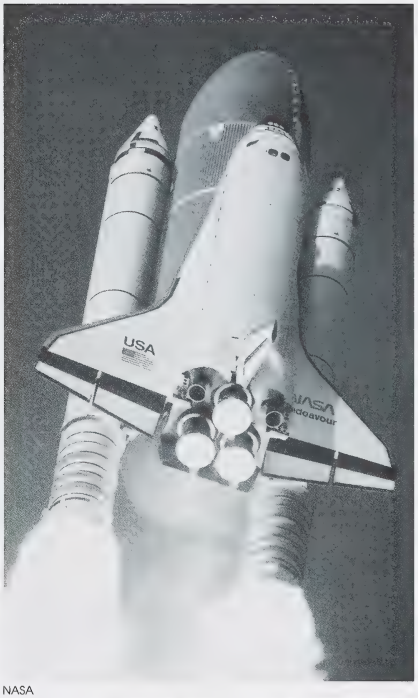
1. $3^x = 2$

2. $4^{x-3} = 8$

Turn to page 96 to compare your responses with those in the Appendix.

Conclusion

In this section, you investigated exponential functions and equations. You analysed the graphs of exponential functions, used technology to determine the exponential functions that best fit a given set of data, and modelled exponential growth and decay. You also explored techniques for solving exponential equations, particularly those equations arising from growth and decay problems.



NASA

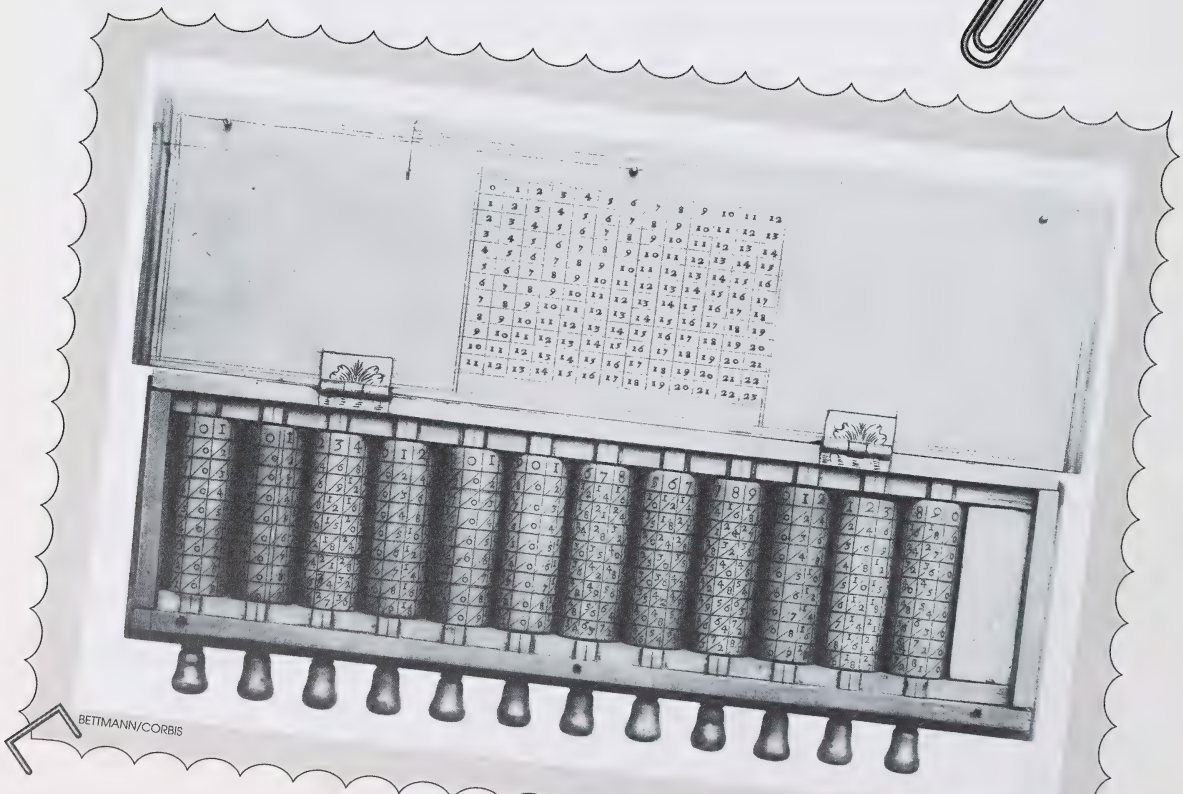
The term *exponential growth* has become part of our everyday language, exemplifying the rapid change in the world today. The last century was a period of unprecedented technological advances, powered flight being only one example.

Langley's steam-powered aircraft of the late nineteenth century are all but forgotten, quickly overshadowed by the achievements of others, such as the Wright brothers at Kittyhawk in 1903, Alexander Graham Bell at Baddeck, Nova Scotia, in 1907, and Igor Sikorsky (the father of the helicopter) in Russia in 1910. By the end of the twentieth century, supersonic flight, rockets and satellites, moon missions, space stations, space shuttles, and interplanetary probes were taken for granted. Do you think exponential growth will continue to be an appropriate term to describe the changes in the twenty-first century?

Assignment

Turn to Assignment Booklet 2A and complete the assignment for Section 1. Remember to submit your completed Assignment Booklet for assessment before starting Section 2.

Section 2



BETTMANN/CORBIS

Logarithmic Functions

Have you thought what it would be like to do complicated arithmetic calculations without the aid of a modern computer or calculator? The calculating machine shown in the photograph is an invention of John Napier (1550–1617), Laird of Merchiston. Mathematician, physicist, engineer, astronomer, and theologian, Napier's greatest contribution is his system of logarithms, which reduced the tedium of complex routines to simple calculations by using prepared tables. Logarithms led to the invention of the slide rule, which, until just a few decades ago, was the indispensable companion of mathematicians, engineers, and scientists alike. Today, a logarithmic function is a powerful tool for solving problems and modelling real-world phenomena.

In this section, you will explore logarithmic functions and their graphs and examine the relationships among exponents and logarithms. You will then investigate the laws of logarithms and how you can employ them to solve equations. Throughout your exploration, you will be introduced to a variety of logarithmic applications, from measuring the intensity of sound to describing acids and bases.

Activity 1: Logarithms

The aerial photograph of earthquake fissures in Western Australia reveals the unbridled power of nature. In a large, uninhabited region, property damage and personal injury is limited; however, in urban or densely populated areas, earthquakes of the same intensity have inflicted enormous damage and numerous casualties.

HULTON-DEUTSCH COLLECTION/CORBIS

In 1935, Charles F. Richter, of the California Institute of Technology, invented a scale for comparing the intensities of earthquakes. The largest earthquake it recorded in the twentieth century occurred in 1960 in Chile. This earthquake had a magnitude 9.5 on the Richter scale.

If you have access to the Internet, you can find out more about the Richter scale and earthquakes at the following website:

http://wwwneic.cr.usgs.gov/neis/general/handouts/richter.html

The Richter scale is an example of a logarithmic scale. In this activity, you will explore logarithms.

Turn to page 94 of *MATHPOWER™ 12* and read "Logarithms."

1. Answer the following questions on page 94 of the textbook:

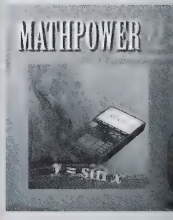
a. "Explore: Rewrite in Another Form"
b. questions 1 to 6 of "Inquire"

Turn to page 98 to compare your responses with those in the Appendix.

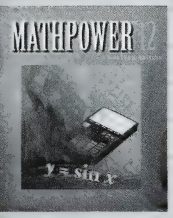
You have just discovered that the inverse of the exponential function $y = b^x$ is the logarithmic function $y = \log_b x$. Recall, from Section 1, that for the exponential function $y = b^x$, the restrictions $b > 0$ and $b \neq 1$ apply. These restrictions also apply to $y = \log_b x$. Remember, logarithms are exponents: $y = \log_b x$ means $x = b^y$.

Mathematical Process

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization



the exponent to which it is necessary to raise a positive base in order to represent a given number



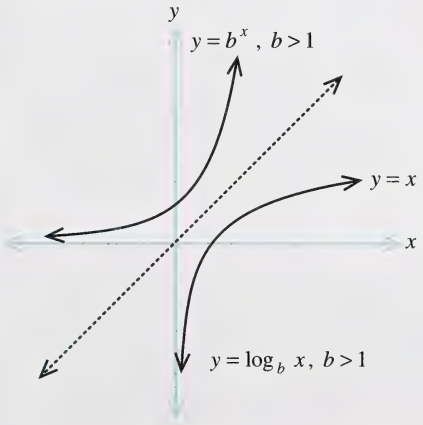
Turn to pages 95 and 96 of *MATHPOWER™ 12* and work through Examples 1 to 3.

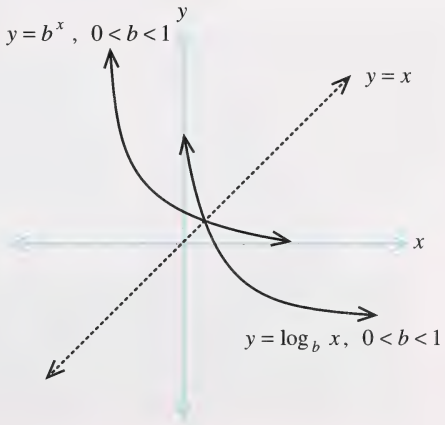
2. Answer the following questions on pages 98 and 99 of the textbook:

a. questions 2, 4, 5, 6, 10, 11, 13, 15, 19, 21, 23, 29, 32, 33, 39, and 40 of "Practice"
b. questions 41 to 43 of "Applications and Problem Solving"

Turn to page 99 to compare your responses with those in the Appendix.

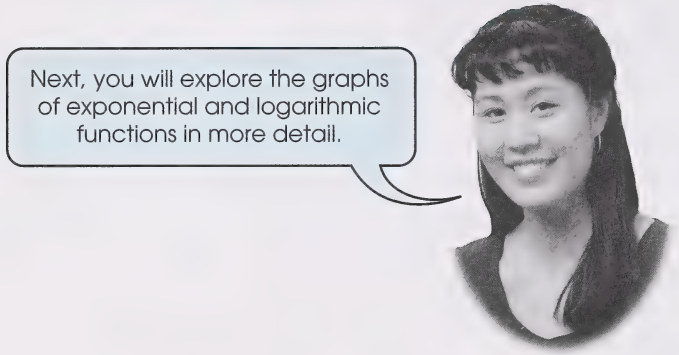
Because the functions $y = b^x$ and $y = \log_b x$, where $b > 0$ and $b \neq 1$, are inverses, their graphs are mirror images of each other in the line $y = x$.





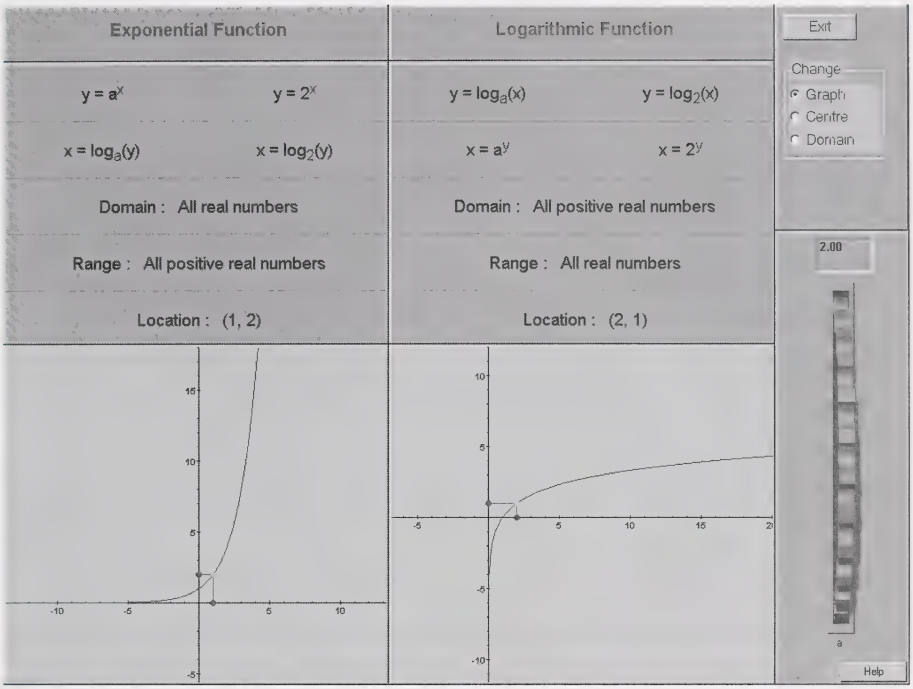
The following table applies to both pairs of graphs.

Graph	$y = b^x$	$y = \log_b x$
Domain	Reals	$x > 0$
Range	$y > 0$	Reals
***x*-intercept**	none	1
***y*-intercept**	1	none
Horizontal Asymptote	$y = 0$	none
Vertical Asymptote	none	$x = 0$

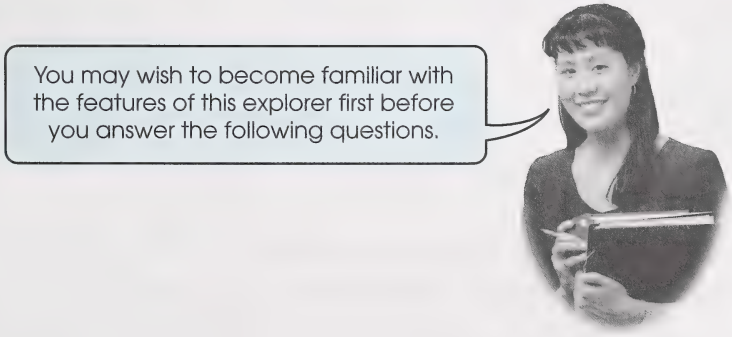


From the Pure Mathematics 30 Companion CD, open *The Logarithmic-Exponential Explorer* from the Explorers folder. This explorer allows you to examine, simultaneously, the graphs of an exponential function and the corresponding logarithmic function with the same base. **Note:** Before you use this explorer, you should open and read the file titled *README.TXT*. This file contains important information about the Pure Mathematics 30 Companion CD.

Note: The graphics and procedures that follow are from the *Windows* version of the explorer. However, the graphics and procedures from the Macintosh version are similar.



The base is designated by the variable *a*. You can change the value of *a* for both graphs at the same time by using the wheel on the right. Simply click and drag the wheel upward or downward, or click in the rectangle above the wheel and enter the value manually. Directions for using this explorer can be found by either clicking on the Help button located in the lower-right corner of the screen or by right-clicking anywhere on the graph and choosing "Help."

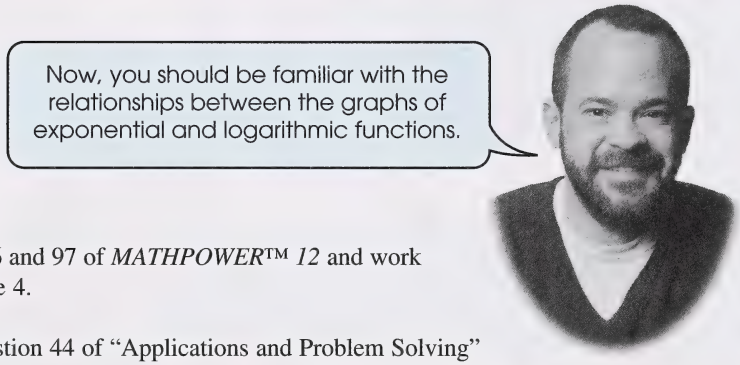


3. Use "The Logarithmic-Exponential Explorer" to complete the following table.

Graph	$y = 3^x$	$y = 0.2^x$	$y = \log_3 x$	$y = \log_{0.2} x$
Domain				
Range				
***x*-intercept**				
***y*-intercept**				
Horizontal Asymptote				
Vertical Asymptote				

4. Explain how changing the base changes the graph of a logarithmic function.

Turn to page 102 to compare your responses with those in the Appendix.



Turn to pages 96 and 97 of *MATHPOWER™ 12* and work through Example 4.

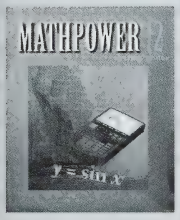
5. Answer question 44 of "Applications and Problem Solving" on page 99 of the textbook.

Turn to page 103 to compare your response with the one in the Appendix.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

Up to this point, you have studied the logarithmic function from a purely theoretical perspective. Next, you will examine how you can use the logarithmic function to answer practical problems, starting with the Richter scale.



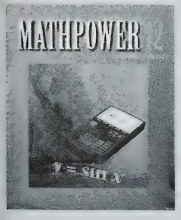
Turn to pages 97 and 98 of *MATHPOWER™ 12* and work through Examples 5 and 6.

6. Answer questions 46, 49, 50.a., and 55 of "Applications and Problem Solving" on pages 99 and 100 of the textbook.

Turn to page 105 to compare your responses with those in the Appendix.

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization



Now Try This

Many practical applications of exponential and logarithmic functions involve change—either growth or decay. Even the rate at which hot chocolate cools in a cup can be modelled using a logarithmic function.

Now, here is another kind of change!

7. Turn to page 100 of *MATHPOWER™ 12* and answer "WORD POWER."

Turn to page 108 to compare your response with the one in the Appendix.

Looking Back

In this activity, you were introduced to the logarithmic function. You discovered that the logarithmic function and the exponential function are inverses. You applied this definition and, with the assistance of technology, analysed the graphs of a variety of exponential and logarithmic functions. Lastly, you explored several practical applications of logarithms.

If you have access to the Internet, research applications of logarithms and record your findings in your journal. Exchange your research results with another student taking Pure Mathematics 30. Add the applications you missed to your journal entries.

Mathematical Process
- [x] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

Activity 2: Laws of Logarithms

John Napier (1550–1617)—Scottish lord, astronomer, and amateur mathematician—invented his system of logarithms as a way of circumventing tedious, time-consuming, and error-prone arithmetic calculations, particularly those involving multiplications, divisions, powers, and roots. In 1625, his collaboration with an English mathematician by the name of Briggs led to the preparation of base-10 logarithmic tables, variations of which were commonly used until the invention of the calculator and computer.

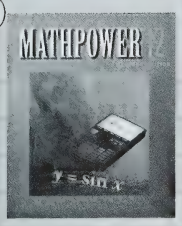
Logarithms provided the astronomer Johannes Kepler with the mathematical tools to formulate his third law from observational data. Kepler's Third Law, which deals with the periods and orbits of the planets, was a stepping stone to Newton's development of his theory of gravitation.

Mathematical Process
- [ ] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

If you have access to the Internet, you can find out more about John Napier at the following website:

http://www-groups.dcs.st-and.ac.uk:80/~history/Mathematicians/Napier.html

In this activity, you will examine the laws of logarithms. These laws were once used the way calculators and computers are used today—as aids for computations. Today, these laws of logarithms are important tools in solving practical problems involving exponents and exponential expressions.



Turn to page 102 of *MATHPOWER™ 12* and read "Laws of Logarithms."

1. Answer the following questions on page 102 of the textbook:

a. questions a. to c. of "Explore: Use a Model"
b. questions 1 to 5 of "Inquire"

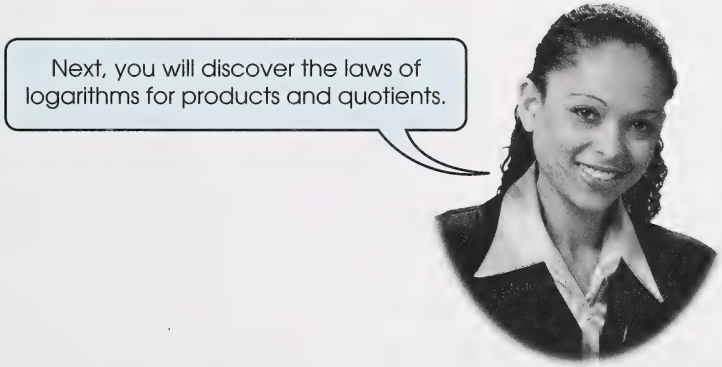
Turn to page 108 to compare your responses with those in the Appendix.

Mathematical Process

- Communication
- [X] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

In Activity 1, you discovered that logarithms are exponents. So, it makes sense that the laws of logarithms should parallel the laws of exponents. Recall that when you multiply and divide powers with the same base, you add and subtract the exponents.

$$b^m \times b^n = b^{m+n} \qquad \frac{b^m}{b^n} = b^{m-n}$$

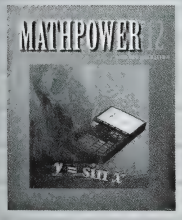


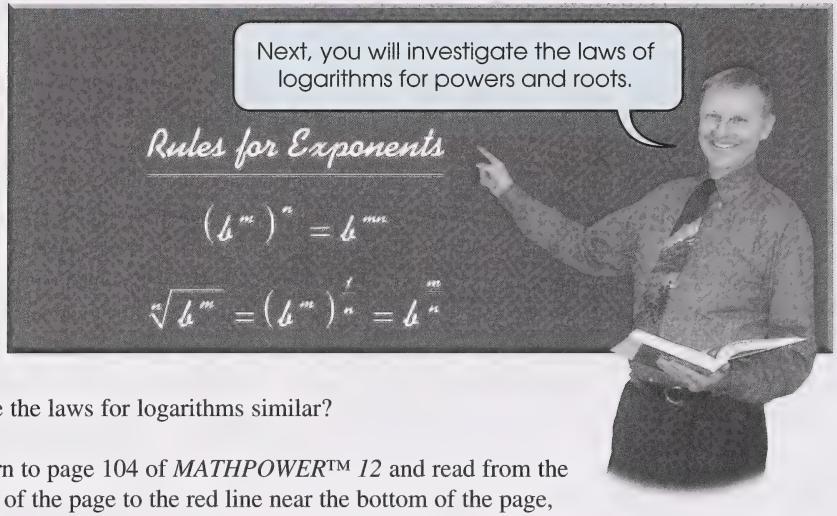
Read page 103 of *MATHPOWER™ 12*, working through Example 1.

2. Answer the following questions on page 106 of the textbook:

a. questions 1, 3, 5, 7, 19, and 21 of "Practice"
b. questions 43, 44.a., and 45 of "Applications and Problem Solving"

Turn to page 110 to compare your responses with those in the Appendix.




Next, you will investigate the laws of logarithms for powers and roots.
Rules for Exponents
$(b^m)^n = b^{mn}$
$\sqrt[n]{b^m} = (b^m)^{\frac{1}{n}} = b^{\frac{m}{n}}$

Are the laws for logarithms similar?

Turn to page 104 of *MATHPOWER™ 12* and read from the top of the page to the red line near the bottom of the page, working through Example 2.

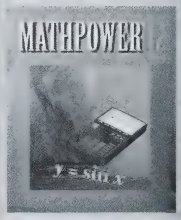
3. Answer the following questions on page 106 of the textbook:

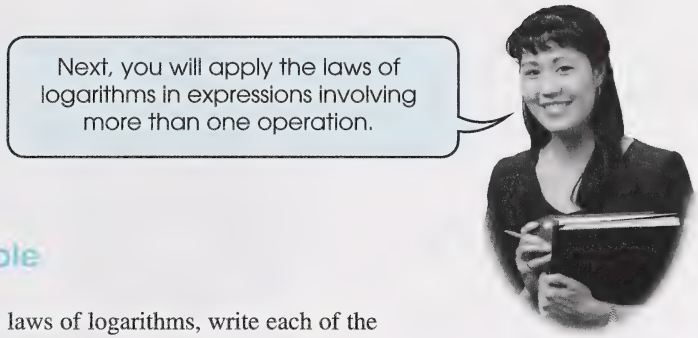
a. questions 11, 13, 16, 17, 22, and 26 of "Practice"
b. questions 44.b., and 46 of "Applications and Problem Solving"

Turn to page 112 to compare your responses with those in the Appendix.

Mathematical Process

- Communication
- [X] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization





Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- [x] Reasoning
- Technology
- Visualization

Example

Using the laws of logarithms, write each of the following expressions in expanded form:

a. $\log_b \dfrac{p^3\sqrt{q}}{rs^2}$

b. $\log_b \dfrac{p}{q^3 b}$

Solution

a. $\log_b \dfrac{p^3\sqrt{q}}{rs^2} = \log_b\left(p^3\sqrt{q}\right) - \log_b\left(rs^2\right)$ ← Quotient of Logarithms Law

$= \left(\log_b p^3 + \log_b \sqrt{q}\right) - \left(\log_b r + \log_b s^2\right)$ ← Product Law

$= \left(3\log_b p + \frac{1}{2}\log_b q\right) - \left(\log_b r + 2\log_b s\right)$ ← a Root Law

$= 3\log_b p + \frac{1}{2}\log_b q - \log_b r - 2\log_b s$ ← Remove the parentheses.

b. $\log_b \dfrac{p}{q^3 b} = \log_b p - \log_b\left(q^3 b\right)$ ← Logarithm of a Quotient Law

$= \log_b p - \left(\log_b q^3 + \log_b b\right)$ ← Logarithm of a Product Law

$= \log_b p - \left(3\log_b q + \log_b b\right)$ ← Logarithm of a Power Law

$= \log_b p - \left(3\log_b q + 1\right)$ ← definition of a logarithm

$= \log_b p - 3\log_b q - 1$ ← Remove the parentheses.

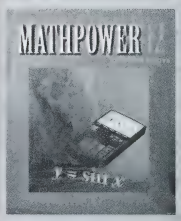
Turn to pages 104 and 105 of *MATHPOWER™ 12* and work through Examples 3 and 4.

4. Answer the following questions on pages 106 and 107 of the textbook:

a. questions 28, 31, 33, 36, and 42 of "Practice"
b. questions 44.c. and 52 of "Applications and Problem Solving"

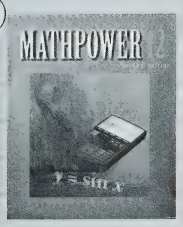
Turn to page 113 to compare your responses with those in the Appendix.

Now that you are familiar with the laws of logarithms, you will apply them to solve practical problems. One application involves the pH of a solution, the logarithmic scale that shows how acidic or basic a solution is. The pH of pure water is 7, the dividing line between acids and bases.



Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization



Turn to page 105 of *MATHPOWER™ 12* and read from the red line to the bottom of the page, working through Example 5.

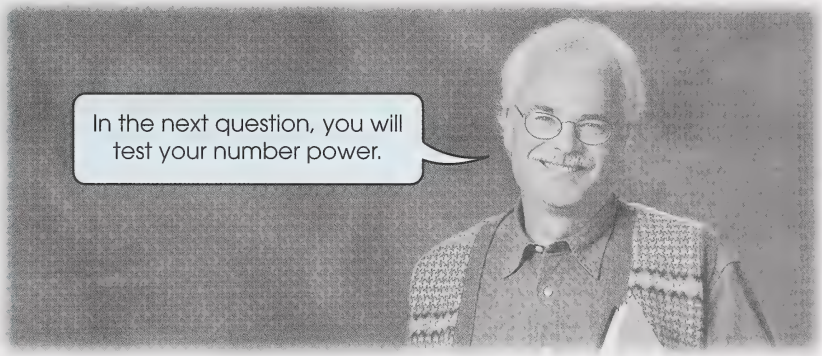
5. Answer questions 47.b., 47.c., 48, and 50 of "Applications and Problem Solving" on page 107 of the textbook.

Turn to page 115 to compare your responses with those in the Appendix.

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- Reasoning
- Technology
- Visualization

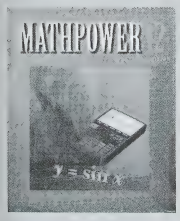
Now Try This

The power of logarithms lies in the ease in which they can be applied when translating complicated expressions into simpler expressions, making them easier to evaluate or manipulate.



6. Turn to page 107 of *MATHPOWER™ 12* and answer "NUMBER POWER."

Turn to page 116 to compare your response with the one in the Appendix.



Mathematical Process
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Looking Back

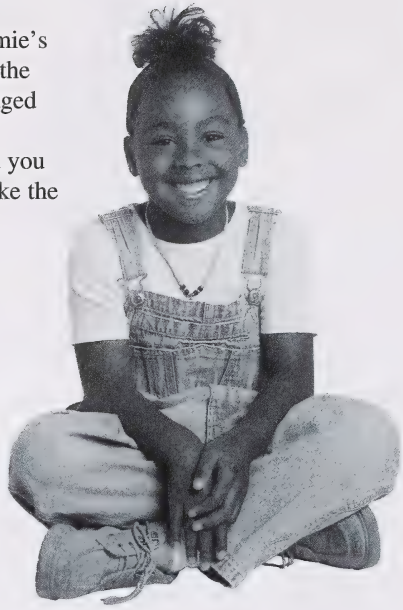
In this activity, you investigated the laws of logarithms: Logarithm of a Product Law, Logarithm of a Quotient Law, Logarithm of a Power Law, and Logarithm of a Root Law. You examined how to apply these laws in combination and to solve real-world problems.

In your journal, discuss how the laws of logarithms are related to the laws of exponents. Use examples to illustrate your answer.

Activity 3: Equation Solving

Mathematical Process
- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Jamie's parents intend to deposit $10 000 into a Registered Education Savings Plan (RESP) for Jamie's college education. They are considering investing the money in a mutual fund that, in the past, has averaged 10% per annum. Jamie's parents would like the investment to grow to at least $40 000. How could you help them determine the number of years it will take the investment to grow to that amount?



Use the formula for compound interest, $A(t) = P(1+i)^n$, where $A(t) = 40\,000$, $P = 10\,000$, $i = 9\% = 0.09$, and $n = t$.

$$A(t) = P(1+i)^n$$
$$40\,000 = 10\,000(1+0.10)^t$$
$$4 = (1.10)^t$$

You could find the number of years, t, using trial and error.

Try $t = 10$.

$(1.10)^{10} \doteq 2.59$

Try $t = 13$.

$(1.10)^{13} \doteq 3.45$

Try $t = 14$.

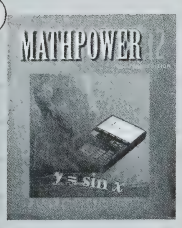
$(1.10)^{14} \doteq 3.80$

Try $t = 15$.

$(1.10)^{15} \doteq 4.18$

It will take the investment 15 years to grow to at least $40 000.

A more efficient approach to solving exponential equations, such as $4 = (1.10)^t$, involves the use of logarithms. In this activity, you will discover how to use logarithms to solve equations and verify statements.



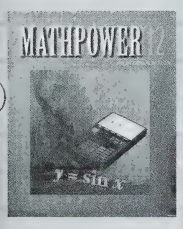
Turn to page 108 of *MATHPOWER™ 12* and read "Logarithms and Equation Solving."

1. Answer the following questions on page 108 of the textbook:

a. "Explore: Solve an Equation"
b. questions 1 to 5 of "Inquire"

Turn to page 116 to compare your responses with those in the Appendix.

Now that you have investigated the limitations of trial and error as an approach to solving exponential equations, you will explore using logarithms as an alternative.



Turn to page 108 of *MATHPOWER™ 12* and read from the red line to the red line on page 109, working through Example 1.

2. Answer questions 1, 2, 4, 9, and 13 of "Practice" on page 113 of the textbook.

Turn to page 118 to compare your responses with those in the Appendix.

In the preceding questions, you solved exponential equations algebraically. You could have solved these equations graphically using your graphing calculator.

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- [x] Visualization

Example

a. Solve $2^{x+1} = 3^{x-1}$ algebraically. Express your answer as an exact value first; then round your answer to 2 decimal places.

b. Solve $2^{x+1} = 3^{x-1}$ graphically. Round your answer to 2 decimal places.

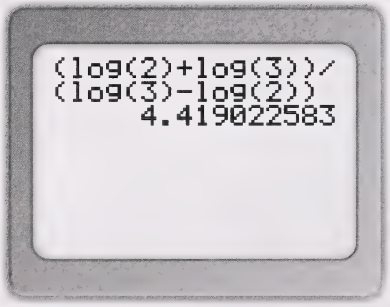
Solution

a.

$$2^{x+1} = 3^{x-1}$$
$$\log 2^{x+1} = \log 3^{x-1}$$
$$(x+1)\log 2 = (x-1)\log 3$$
$$x \log 2 + \log 2 = x \log 3 - \log 3$$
$$\log 2 + \log 3 = x \log 3 - x \log 2$$
$$\log 2 + \log 3 = x(\log 3 - \log 2)$$
$$x = \frac{\log 2 + \log 3}{\log 3 - \log 2}$$

This is the exact value of x!

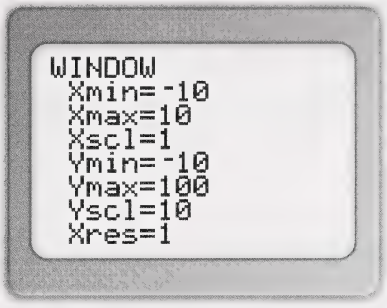
Now, use your graphing calculator to evaluate this expression correct to 2 decimal places. **Note:** Use parentheses for the numerator and denominator.



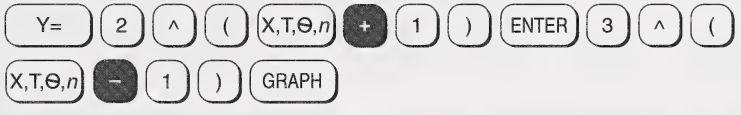
Therefore, $x \doteq 4.42$.

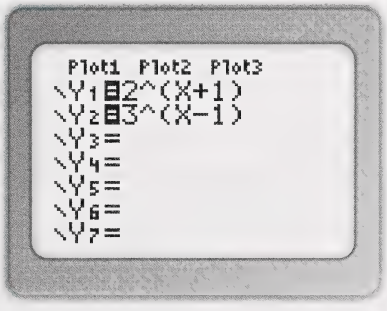
b. To solve $2^{x+1} = 3^{x-1}$ graphically, you need to determine where the graphs of $y = 2^{x+1}$ and $y = 3^{x-1}$ cross. The solution of the original equation is the x-coordinate of the intersection point.

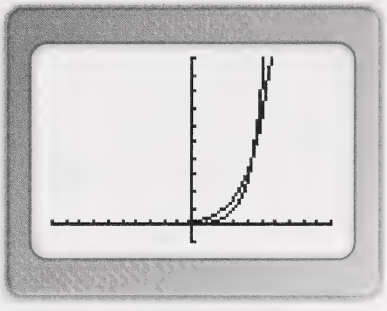
Select an appropriate viewing window. Remember, because you are working with exponential functions, the y-values increase quickly. Use the following window settings.



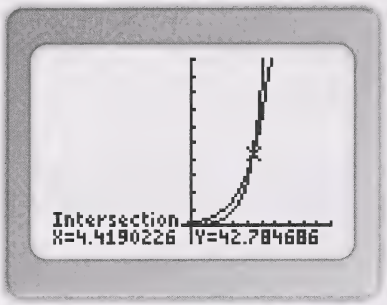
Now, graph $y = 2^{x+1}$ and $y = 3^{x-1}$.





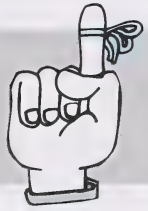


Use the Intersect feature from the CALCULATE menu to locate the point where the two graphs intersect .



Therefore, $x \doteq 4.42$.

Remember: When you solve exponential equations using logarithms, your solutions are exact values. When you solve exponential equations graphically, your solutions are approximate values!



Turn to pages 109 and 110 of *MATHPOWER™ 12* and work through Examples 2 and 3.

3. Answer the following questions on pages 113 to 115 of the textbook:

a. questions 26, 27, 30, and 31 of "Practice"
b. questions 43, 44, 47, and 53 of "Applications and Problem Solving"

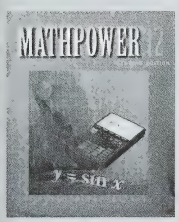
Turn to page 120 to compare your responses with those in the Appendix.

Up to this point, all your work has involved base-10 logarithms, or common logarithms. On your graphing calculator, common logarithms are obtained using [LOG]. There is another logarithm key on your calculator; it is [LN]. This key is used for finding natural logarithms, or base-e logarithms, where $e = 2.718\,281\,828\ldots$. It is an irrational number, like π. If you take Mathematics 31 or any other calculus course, you will use natural logarithms extensively in your work.

a base-10 logarithm

a base-e logarithm

In Activity 1 of this section, you discovered that any positive number other than 1 can be used as the base of a logarithm. How can you evaluate these logarithms if your calculator only has two logarithm keys?



Turn to page 111 of *MATHPOWER™ 12* and read from the top of the page to the red line near the bottom of the page, working through Example 4.

4. Answer the following questions on pages 113 and 114 of the textbook:

a. questions 15 and 19 of "Practice"
b. question 41 "Applications and Problem Solving"

Turn to page 125 to compare your responses with those in the Appendix.

When solving equations, look for a finite number of solutions. For example, the roots of $x^2 - 4 = 0$ are ± 2. No other number will satisfy this equation.

However, there are statements of equality that are true for an infinite number of values of the variable. As a matter of fact, they are true for all values of the variable for which each part of the equality is defined. Such statements are called identities. For example, the statement $\frac{4x^2}{4x} = x$ is an identity. This statement is true for all real values of x except $x = 0$, which makes the left side of the equation undefined.

a statement of equality true for all values of the variable for which each part of the equality is defined

You will now investigate **logarithmic identities**.

an identity involving a logarithmic function

Example

Prove the following identity:

$$\log_b \frac{1}{x} = -\log_b x, \text{ for all } x > 0$$

Solution

You must show that the left side of the identity is equal to the right side of the identity.

LS	RS
$\log_b \frac{1}{x}$	$-\log_b x$
$= \log_b 1 - \log_b x$	
$= \log_b b^0 - \log_b x$	
$= 0 - \log_b x$	
$= -\log_b x$	

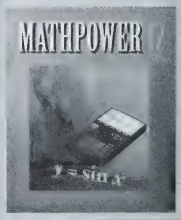
LS = RS

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- [x] Reasoning
- Technology
- Visualization

Next, you will encounter identities for which you must first convert logarithmic statements into exponential statements.

Turn to page 111 of *MATHPOWER™ 12* and read from the red line near the bottom of the page to the red line on page 112, working through Example 5.



5. Answer the following questions on pages 113 to 115 of the textbook:

a. questions 33.b. and 35.b. of "Practice"
b. questions 48 and 51 of "Applications and Problem Solving"

Turn to page 127 to compare your responses with those in the Appendix.

an equation in which the variable occurs as part of a logarithm

Example

Solve $\log x + \log(x-9) = 1$.

Solution

The trick is to rewrite the two logarithms as a single logarithm and then convert the resulting statement from logarithmic to exponential form. The exponential equation will then be easier to solve.

Caution: Be careful when combining logarithms. You must keep track of any restrictions on the logarithms in the original equation.

Here, $\log x$ is defined only if $x > 0$ and $\log(x-9)$ is defined only if $x > 9$. Therefore, you must reject any answer less than or equal to 9.

$$\log x + \log(x-9) = 1$$
$$\log x(x-9) = 1$$
$$x(x-9) = 10^1$$
$$x^2 - 9x = 10$$
$$x^2 - 9x - 10 = 0$$
$$(x+1)(x-10) = 0$$

$\therefore x+1=0$ or $x-10=0$

$x=-1$ $\quad$ $x=10$

Recall that the restriction on the variable is $x > 9$. Therefore, the solution is $x = 10$.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- [x] Reasoning
- Technology
- Visualization

Turn to pages 112 and 113 of *MATHPOWER™ 12* and work through Example 6.

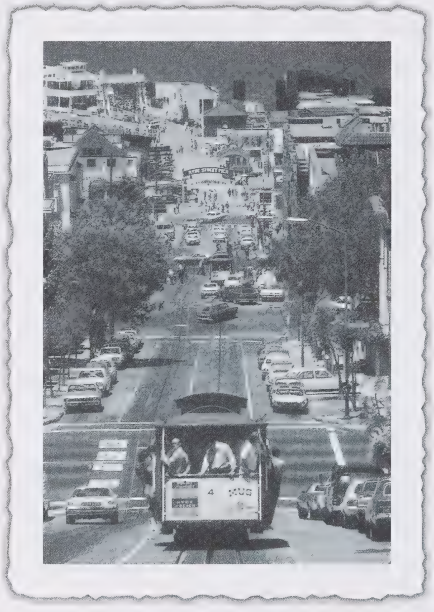
6. Answer the following questions on pages 113 to 115 of the textbook:

a. questions 21 and 24 of "Practice"
b. questions 40.a., 40.c., 40.f., and 62.c. of "Applications and Problem Solving"

Turn to page 128 to compare your responses with those in the Appendix.

Now Try This

In this activity, you have encountered two logarithmic scales: the Richter scale and the pH scale. Another logarithmic scale you may have heard of is the decibel scale. It is used for measuring and comparing the intensities of sound.



Many cities, like San Francisco, have noise bylaws that refer to the decibel scale when setting legal limits. In the following investigations, you will work with the definition and application of the decibel scale.

7. Turn to "CONNECTING MATH AND PHYSICS" on pages 124 and 125 of *MATHPOWER™ 12* and answer the following:

a. questions 1 to 6 of Investigation 1, "Comparing the Intensity of Sounds"

b. questions 1 and 2 of Investigation 2, "The Effect of Combining Sounds"

Turn to page 131 to compare your responses with those in the Appendix.

Mathematical Process

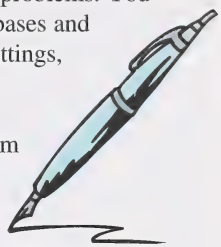
- [ ] Communication
- [x] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

Mathematical Process

- [x] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

Looking Back

In this activity, you first examined procedures for solving exponential equations using logarithms; then you applied these procedures to solving real-world problems. You also explored techniques of evaluating logarithms with a variety of bases and investigated logarithmic equations and identities. In each of these settings, you applied the rules for logarithms.



In your journal, outline the steps for solving each equation or problem type; then select an example of each type. Exchange journals with another student taking Pure Mathematics 30. Together, suggest improvements that can be made to each other's outlines.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You must have a thorough understanding of the laws and properties of logarithms to be successful when using logarithms to solve equations or model practical problems. The following is a list of the laws and properties that you studied in this section.

Logarithmic Laws and Properties

- Definition of a Logarithm

  $y = \log_b x$, where $x > 0$, means $x = b^y$

- Relationship Between the Logarithmic and Exponential Function

  The logarithmic function $y = \log_b x$ or $x = b^y$ is the inverse of the exponential function $y = b^x$.

- $\log_b b^n = n$

- $\log_b 1 = 0$ because $1 = b^0$.

Remember:
The restrictions on the base, b, are $b > 0$ and $b \neq 1$.

- Logarithm of a Product Law

$$\log_b xy = \log_b x + \log_b y$$

- Logarithm of a Quotient Law

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

- Logarithm of a Power Law

$$\log_b x^n = n \log_b x$$

- Logarithm of a Root Law

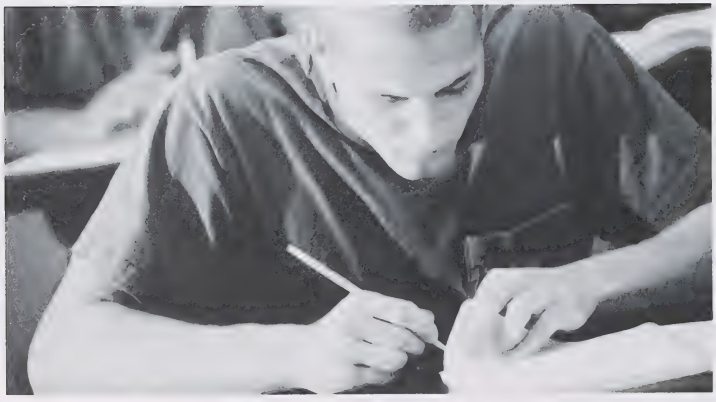
$$\log_b \sqrt[n]{x} = \frac{1}{n} \log_b x$$

- Common (or Base-10) Logarithms

$\log x$ means $\log_{10} x$

- Change of Base Formulas

general: $\log_b a = \dfrac{\log_c a}{\log_c b}$ base-10: $\log_b a = \dfrac{\log a}{\log b}$



Turn to pages 126 to 128 of *MATHPOWER™ 12* and answer the following:

1. questions 16, 21, 22, 27, 33, 37, and 41 of "Review"
2. questions 6.a., 8.a., 9.b., and 10.b. of "Chapter Check"

Turn to page 134 to compare your responses with those in the Appendix.

Enrichment

If you began walking at 1 km/h and then doubled your speed over a one-minute interval, you would be walking at 2 km/h. But suppose you increased your speed by 50% every half-minute. How fast would you be walking at the end of one minute?

At the end of the first half-minute, you would be walking 1.5 times faster or $1(1.5) = 1.5$ km/h. At the end of the second half-minute, you would have increased your speed by 1.5 times again. Therefore, your final speed would be $(1.5)(1.5) = 1.5^2 = 2.25$ km/h.

Suppose you increased your speed by 25% every quarter minute. What would your speed be at the end of one minute? Remember, your speed would be 1.25 times faster every quarter minute.

Time Elapsed (s)	0	15	30	45	60
Speed (km/h)	1	1.25	$(1.25)^2$	$(1.25)^3$	$(1.25)^4$

In this case, your speed would be $(1.25)^4 \doteq 2.441$ km/h after the first minute.

Suppose you increased your speed 10 times a minute by 10% $\left(\text{or } \frac{1}{10}\right)$. Your final speed at the end of one minute would be $\left(1+\frac{1}{10}\right)^{10} \doteq 2.594$ km/h.

Consider what would happen if this process were continued indefinitely.

$$(1+1)^1 = 2$$

$$\left(1+\frac{1}{10}\right)^{10} = 2.593\,742$$

$$\left(1+\frac{1}{100}\right)^{100} = 2.704\,814$$

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.716\,924$$

$$\left(1+\frac{1}{10\,000}\right)^{10\,000} = 2.718\,146$$

$$\left(1+\frac{1}{100\,000}\right)^{100\,000} = 2.718\,268$$

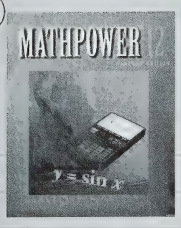
$$\left(1+\frac{1}{1\,000\,000}\right)^{1\,000\,000} = 2.718\,280$$

$$\left(1+\frac{1}{10\,000\,000}\right)^{10\,000\,000} = 2.718\,282$$

Answers are rounded to 6 decimal places.

It turns out that the expression $\left(1+\frac{1}{n}\right)^n$ approaches 2.718 281 828... as n approaches infinity. This is the value of e, the base of the natural logarithm.

On your graphing calculator, the natural logarithm is marked LN. The base e is used in exponential functions that model continuous growth and decay.

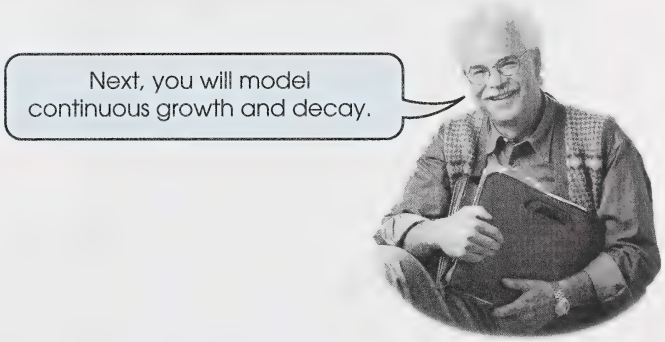


Turn to page 116 of *MATHPOWER™ 12* and read "Continuous Exponential Growth or Decay."

1. Answer the following questions on page 116 of the textbook:

a. "Explore: Use a Table"
b. questions 1 to 5 of "Inquire"

Turn to page 138 to compare your responses with those in the Appendix.



Turn to page 117 of *MATHPOWER™ 12* and read from the top of the page to the bottom of page 118, working through Examples 1 and 2.

2. Answer the following questions on pages 119 and 120 of the textbook:

a. questions 1, 3, 7, 13, 15, and 25 of "Practice"
b. questions 26, 27, and 32 of "Applications and Problem Solving"

Turn to page 139 to compare your responses with those in the Appendix.

Conclusion

In this section, you explored logarithmic functions, their applications, and their graphs. You examined the relationships among exponents and logarithms and between the exponential function and the logarithmic function. Also, you investigated the laws of logarithms and used these laws to solve equations. You then solved exponential equations and logarithmic equations and proved logarithmic identities. Throughout these explorations, you solved a variety of practical problems, ranging from the intensity of earthquakes to the description of the concentration of acids and bases using the pH scale.

GENEVIEVE NAYLOR/CORBIS

One of the early applications of the logarithms that arose from the work of Napier and Briggs was the invention of the slide rule in the seventeenth century. Until the development of inexpensive calculators and computers just a few decades ago, the slide rule was the indispensable companion of scientists, engineers, and mathematicians alike.

In your future study of mathematics, particularly if you pursue calculus, you will discover the power and practicality of both common and natural logarithms.

Assignment

Turn to Assignment Booklet 2B and complete the assignment for Section 2.

Module Summary

In Section 1, you explored the exponential function and its graph. You practised techniques for solving equations arising from the application of exponential functions to solve practical problems. These problems included modelling population growth, compound interest, depreciation, light absorption, and radioactive decay.

In Section 2, you investigated logarithmic functions. You discovered that logarithmic and exponential functions are inverses and that you can develop the laws for logarithms from the laws of exponents. You then solved a variety of equations, proved identities, and modelled practical problems in growth and decay similar to those introduced in Section 1. You also explored logarithmic scales, such as the pH scale and the Richter scale.

One of the problem situations you modelled in both sections involved the decay of radioactive isotopes. Did you know that Canadian scientists, like Harold Elford Johns, pioneered the use of radioactive isotopes in the treatment of cancer? The "cobalt bomb" (using a radioactive isotope of cobalt), developed at the University of Saskatchewan, revolutionized cancer treatment throughout the world. In 1973, in recognition of his work, H. E. Johns received the prestigious Gairdner Foundation International Award for his work in cobalt and high-energy radiotherapy.

HULTON-DEUTSCH COLLECTION/CORBIS

Final Module Assignment

Turn to Assignment Booklet 2B and complete the final module assignment. Remember to submit your completed Assignment Booklet for assessment.

Glossary

Suggested Answers

Credits

Glossary

doubling time: the time it takes for a population to double

common logarithm: a base-10 logarithm

exponential equation: an equation in which the variable occurs in an exponent

exponential function: a function of the form $y = Ab^x$, where A is a non-zero constant and $b > 0$ and $b \neq 1$

exponential growth: growth that can be modelled by an exponential function

generation time: doubling time

half-life: the time it takes for a radioactive substance to decay to one-half its original amount

identity: a statement of equality true for all values of the variable for which each part of the equality is defined

logarithm: the exponent to which it is necessary to raise a positive base in order to represent a given number

logarithmic equation: an equation in which the variable occurs as part of a logarithm

logarithmic function: a function of the form $y = \log_b x$, where $b \neq 1$ and $b > 0$

logarithmic identity: an identity involving logarithmic functions

natural logarithm: a base-e logarithm

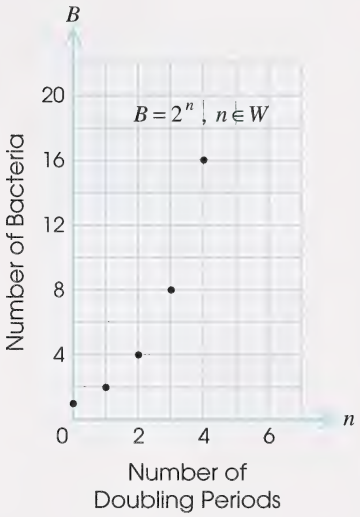
step function: a function whose graph increases in discrete amounts and is constant between increments

Suggested Answers

Section 1: Activity 1

1. a. Textbook question "Explore: Develop a Model," p. 77

Number of Doubling Periods, *n*	Number of Bacteria, *B*
0	1
1	2
2	4
3	8
4	16
5	32
6	64



b. Textbook questions 1 to 7 of "Inquire," pp. 77 and 78

1. An exponential equation relates B and n.

2. In this situation, it does not make sense to join the data points. The number of bacteria is a natural number, and the number of doubling periods is a whole number.

3. In 2 h, there are $\frac{120 \text{ min}}{30 \text{ min}} = 4$ doubling periods.

$\therefore B = 2^n$

$= 2^4$

$= 16$

After 2 h, 16 bacteria result from a single bacterium.

4. If there were 100 bacteria initially, there would be $100 \times 16 = 1600$ bacteria 2 h later. For this sample, the equation would be $B = 100 \times 2^n$.

Section 1: Activity 1 (continued)

5. Number of doubling periods $= \dfrac{\text{time}}{\text{doubling time}}$

$$n = \frac{60t}{30} \quad \text{There are } 60t \text{ minutes in } t \text{ hours.}$$
$$= 2t$$

$\therefore\ B = 500 \times 2^{2t}$

6. To find the number of bacteria that were present one hour before the count was done, use $t = -1$.

$$\therefore\ B = 500 \times 2^{2t}$$
$$= 500 \times 2^{2(-1)}$$
$$= 500 \times 2^{-2}$$
$$= 500 \times \frac{1}{4}$$
$$= 125$$

There were 125 bacteria before the count was done.

7. $B(t) = B_0 \times 2^{\frac{60t}{k}}$, where $B(t)$ is the number of bacteria after t hours and k is the doubling time (in minutes).

2. a. Textbook questions 16 and 18 to 23 of "Practice," p. 82

16. Let $B(t)$ represent the number of bacteria at any time, t (in minutes).

The initial count shows 300 bacteria, and the number is doubling every 20 min.

$$\therefore\ B(t) = 300 \times 2^{\frac{t}{20}}$$
$$B(40) = 300 \times 2^{\frac{40}{20}}$$
$$= 300 \times 2^2$$
$$= 300 \times 4$$
$$= 1200$$

There are 1200 bacteria after 40 min.

18. $2\text{ h} = 2\text{ h} \times 60\text{ min/h}$
$= 120\text{ min}$

$\therefore B(t) = 300 \times 2^{\frac{t}{20}}$
$B(120) = 300 \times 2^{\frac{120}{20}}$
$= 300 \times 2^6$
$= 300 \times 64$
$= 19\,200$

There are 19 200 bacteria after 2 h.

19. Initially, $t = 0$ h.

$\therefore N(t) = 9000 \times 2^{\frac{t}{4}}$
$N(0) = 9000 \times 2^{\frac{0}{4}}$
$= 9000 \times 2^0$
$= 9000 \times 1$
$= 9000$

There were 9000 bacteria in the initial count.

20. After the first doubling period, there are $2 \times 9000 = 18\,000$ bacteria.

$\therefore N(t) = 9000 \times 2^{\frac{t}{4}}$
$18\,000 = 9000 \times 2^{\frac{t}{4}}$
$\frac{18\,000}{9000} = \frac{9000 \times 2^{\frac{t}{4}}}{9000}$
$2 = 2^{\frac{t}{4}}$
$2^1 = 2^{\frac{t}{4}}$
$1 = \frac{t}{4}$
$t = 4$

The doubling period is $4 \times 60 = 240$ min.

Section 1: Activity 1 (continued)

21. $N(t) = 9000 \times 2^{\frac{t}{4}}$

$$36\,000 = 9000 \times 2^{\frac{t}{4}}$$

$$\frac{36\,000}{9000} = \frac{9000 \times 2^{\frac{t}{4}}}{9000}$$

$$4 = 2^{\frac{t}{4}}$$

$$2^2 = 2^{\frac{t}{4}}$$

$$2 = \frac{t}{4}$$

$$t = 8$$

There are 36 000 bacteria after 8 h.

22. 1 d = 24 h

$$\therefore N(t) = 9000 \times 2^{\frac{t}{4}}$$

$$N(24) = 9000 \times 2^{\frac{24}{4}}$$

$$= 9000 \times 2^6$$

$$= 9000 \times 64$$

$$= 576\,000$$

There will be 576 000 bacteria one day later.

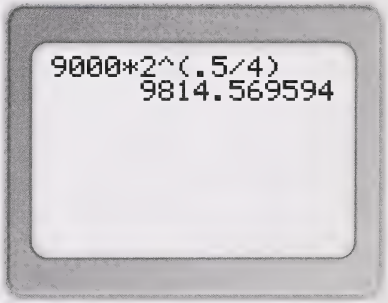
23. 30 min = 0.5 h

$$\therefore N(t) = 9000 \times 2^{\frac{t}{4}}$$

$$N(0.5) = 9000 \times 2^{\frac{0.5}{4}}$$

$$\doteq 9810$$

To the nearest ten, there will be 9810 bacteria present 30 min after the initial count.



b. Textbook question 25 of "Applications and Problem Solving," p. 82

25. a. Let $P(t)$ represent the number of bacteria at any time, t (in minutes).

Because the initial population is 5000 and the number is doubling every 45 minutes,

$$P(t) = 5000(2)^{\frac{t}{45}}$$

b. $P(t) = 5000(2)^{\frac{t}{45}}$

$$\begin{aligned} P(90) &= 5000(2)^{\frac{90}{45}} \\ &= 5000(2)^2 \\ &= 5000(4) \\ &= 20\ 000 \end{aligned}$$

There will be 20 000 bacteria in 90 min.

$$\begin{aligned} P(t) &= 5000(2)^{\frac{t}{45}} \\ P(135) &= 5000(2)^{\frac{135}{45}} \\ &= 5000(2)^3 \\ &= 5000(8) \\ &= 40\ 000 \end{aligned}$$

There will be 40 000 bacteria in 135 min.

$$\begin{aligned} P(t) &= 5000(2)^{\frac{t}{45}} \\ P(180) &= 5000(2)^{\frac{180}{45}} \\ &= 5000(2)^4 \\ &= 5000(16) \\ &= 80\ 000 \end{aligned}$$

There will be 80 000 bacteria in 180 min.

Section 1: Activity 1 (continued)

c. Because the count is for 22.5 min ago, $t = -22.5$.

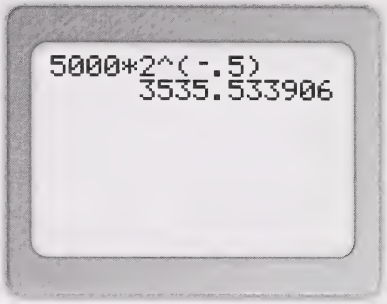
$$\therefore\ P(t) = 5000(2)^{\frac{t}{45}}$$

$$P(-22.5) = 5000(2)^{\frac{-22.5}{45}}$$

$$= 5000(2)^{-0.5}$$

$$\doteq 3536$$

There were approximately 3536 bacteria 22.5 min ago.



d. $10\ \text{h} = 10\ \text{h} \times 60\ \text{min/h}$

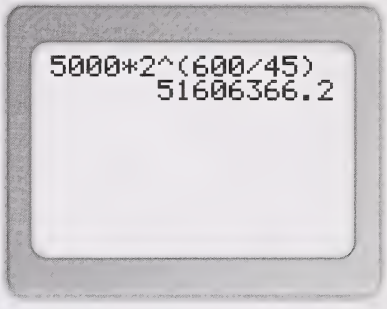
$$= 600\ \text{min}$$

$$\therefore\ P(t) = 5000(2)^{\frac{t}{45}}$$

$$P(600) = 5000(2)^{\frac{600}{45}}$$

$$\doteq 51\,606\,366$$

There will be about 51 606 366 bacteria in 10 h.



e. $3\ \text{h ago} = -3\ \text{h} \times 60\ \text{min/h}$

$$= -180\ \text{min}$$

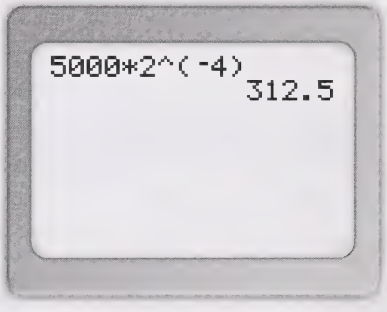
$$\therefore\ P(t) = 5000(2)^{\frac{t}{45}}$$

$$P(-180) = 5000(2)^{\frac{-180}{45}}$$

$$= 5000(2)^{-4}$$

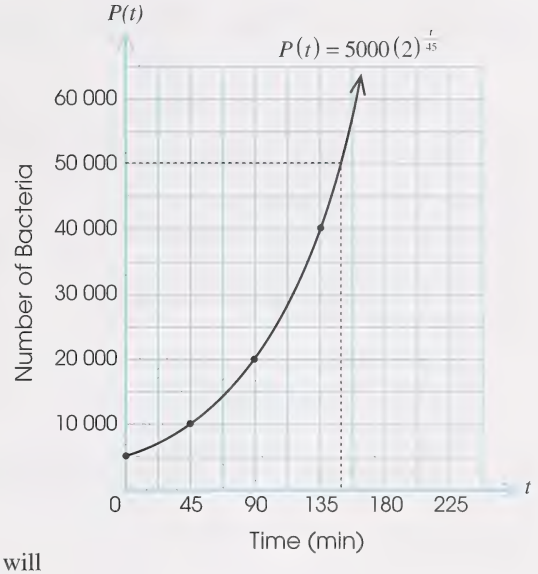
$$\doteq 313$$

There were about 313 bacteria 3 h ago.



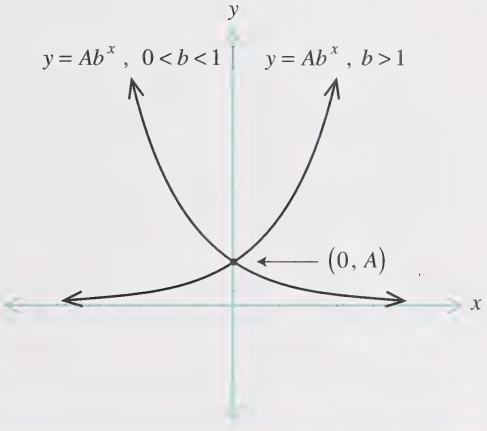
f. Use a table of values to graph $P(t) = 5000(2)^{\frac{t}{45}}$.

t	$P(t)$
0	5000
45	10 000
90	20 000
135	40 000
180	80 000
225	160 000



According to the graph, it will take approximately 150 minutes before there will be about 10 times the initial number of bacteria.

3. a.



Graph	Domain	Range	*x*-intercept	*y*-intercept	Asymptote
$y = Ab^x$, $0 < b < 1$	Reals	$y > 0$	none	$y = A$	$y = 0$
$y = Ab^x$, $b > 1$	Reals	$y > 0$	none	$y = A$	$y = 0$

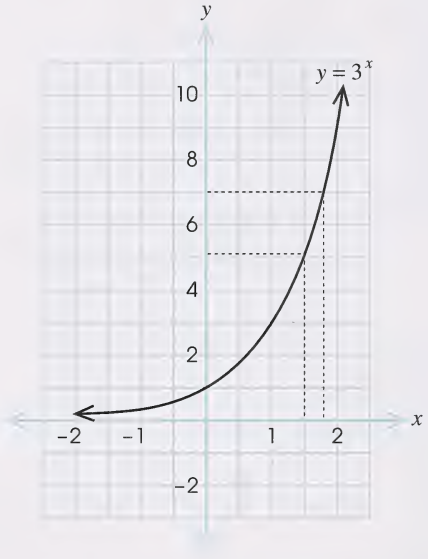
Section 1: Activity 1 (continued)

b. The restriction $b > 0$ is applied so that $y = Ab^x$ is defined for all real values of x. For example, if $b = -2$, then b^x is undefined for values of x such as $\frac{1}{2}$ and $\frac{1}{4}$.

- $(-2)^{\frac{1}{2}} = \sqrt{-2} =$ non-real
- $(-2)^{\frac{1}{4}} = \sqrt[4]{-2} =$ non-real

The restriction $b \neq 1$ is applied so the function remains an exponential function. If $b = 1$, the function $y = Ab^x$ would be $y = A1^x$ or $y = A$. This is a constant function.

4. a. Textbook questions 1, 5, 13, 14, and 15 of "Practice," p. 81



1. From the graph, $3^{1.5} \doteq 5.2$.

5. From the graph, $x \doteq 1.8$.

13. There were 40 bullfrogs at the initial count.

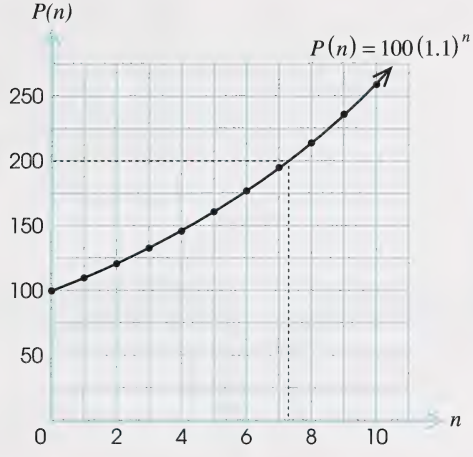
14. It took 5 years for the population to double.

15. There were over 100 bullfrogs after approximately 6.6 years.

b. Textbook questions 24, 26, 27, 28, and 31 of "Applications and Problem Solving," pp. 82 and 83

24.

n	0	1	2	3	4	5	6	7	8	9	10
$P(n)$	100	110	121	133	146	161	177	195	214	236	259



The values of $P(n)$ have been rounded to the nearest whole number.

From the graph, it takes between 7 and 8 years for the population to double.

26. a. Let $P(t)$ represent the country's population at any time, t (in years).

Because the population is increasing 3% annually and the current population is 30 million,

$$P(t) = 30(1+0.03)^t$$
$$= 30(1.03)^t$$

Note: $P(t)$ is in millions of people.

This situation is similar to the compound interest formula, $A(t) = P(1+i)^n$.

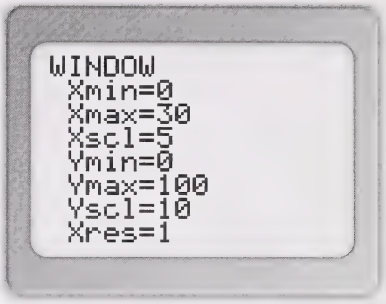
b. In 15 years, $t = 15$.

$$\therefore P(t) = 30(1.03)^t$$
$$P(15) = 30(1.03)^{15}$$
$$\doteq 46.7$$

In 15 years, the population will be approximately 46.7 million.

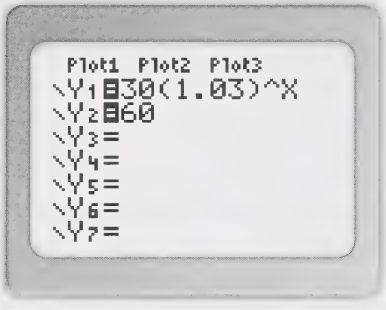
Section 1: Activity 1 (continued)

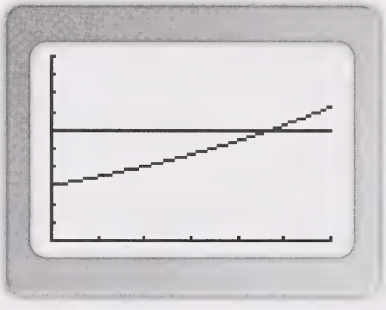
c. There will be 60 million people when the population doubles. Use a graphing calculator to determine how many years it will take for the population to double.



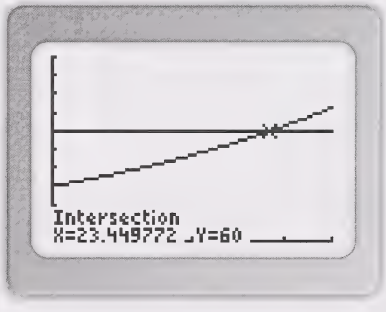
Using the window settings given on the right,

graph $y = 30(1.03)^t$ as Y_1 and $y = 60$ as Y_2.





Use the Intersect feature from the CALCULATE menu to determine where the two graphs intersect.



The population will reach 60 million in approximately 23.4 years.

d. Answers may vary. Factors that could cause this model to break down are changes in the birth rate, changes in the death rate, and changes in immigration.

27. a. Let $P(t)$ represent the community's population after t years.

The population decreases by 13% per year. Therefore, the annual change is -0.13.

Since the present population is 18 000,

$$\begin{aligned} P(t) &= 18\,000(1-0.13)^t \\ &= 18\,000(0.87)^t \end{aligned}$$

b. In 4 years, $t = 4$.

$$\begin{aligned} \therefore\ P(t) &= 18\,000(0.87)^t \\ P(4) &= 18\,000(0.87)^4 \\ &\doteq 10\,300 \end{aligned}$$

In 4 years, the population will be approximately 10 300.

c. When the population is halved, the population will be 9000. Since $P(4) \doteq 10\,300$, try $P(5)$.

$$\begin{aligned} P(5) &= 18\,000(0.87)^5 \\ &\doteq 9000 \end{aligned}$$

The population will be only half the current population in about 5 years.

d. Answers may vary. Factors that could cause this model to change are changes to the rural economy, changes in people's perception of the quality of life in the rural community, and changes in urban areas that either encourages or discourages migration.

28. a. For compound interest, the amount, $A(t)$, is related to the principal, P, by the formula $A(t) = P(1+i)^n$, where t is the time (in years), i is the interest rate per compounding period, and n is the number of compounding periods.

$P = 1700$, $i = 5\% = 0.05$, and $n = t$

$$\begin{aligned} \therefore\ A(t) &= P(1+i)^n \\ &= 1700(1+0.05)^t \\ &= 1700(1.05)^t \end{aligned}$$

Note: Because the interest is compounded annually, $n = t$.

Section 1: Activity 1 (continued)

b. In 6 years, $t = 6$.

$$\therefore A(t) = 1700(1.05)^t$$
$$A(6) = 1700(1.05)^6$$
$$\doteq 2278.16$$

In 6 years, the investment will be worth \$2278.16.

c. Two years ago, $t = -2$.

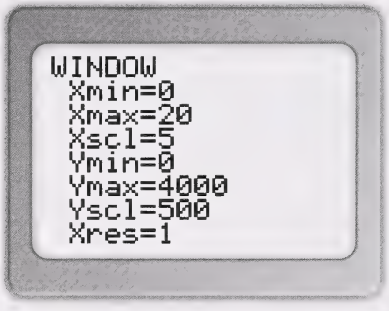
$$\therefore A(t) = 1700(1.05)^t$$
$$A(-2) = 1700(1.05)^{-2}$$
$$\doteq 1541.95$$

Two years ago, the investment was worth \$1541.95.

d. The investment will be worth \$3400 when it doubles.

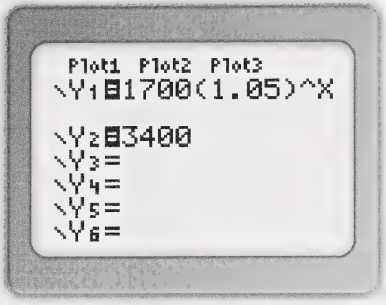
Use a graphing calculator to determine how long it will take the investment to double.

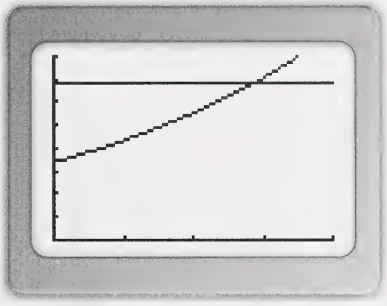
Use the following window settings.



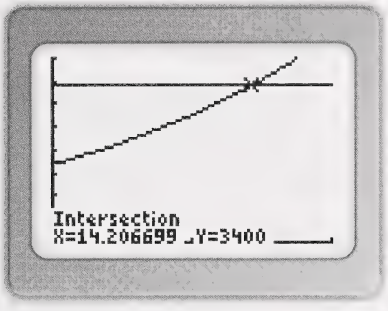
Graph $y = 1700(1.05)^t$ as Y_1 and $y = 3400$ as Y_2.

Y= 1 7 0 0 (1 • 0 5) ^ X,T,Θ,*n* ENTER
3 4 0 0 GRAPH





Use the Intersect feature from the CALCULATE menu to determine where the two graphs intersect.



Because the interest is calculated annually, it will take 15 years for the principal to at least double.

e. Since every dollar of the original deposit must double in value, the time required is independent of the original investment. Therefore, the answer to question 28.d. would not change.

31. a. For the equation $P = P_0\left(b^t\right)$,

P = population at time t

P_0 = present population

$b = 1 + i$, where i is the percent increase or decrease in population

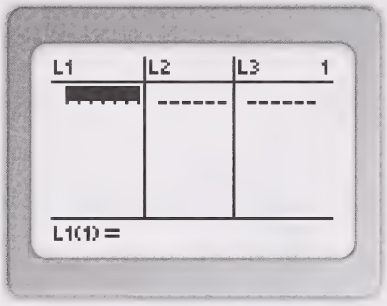
t = time elapsed since the initial count

b. If $b > 1$, the population is increasing. If $0 < b < 1$, the population is decreasing.

Section 1: Activity 1 (continued)

5. Textbook questions 1 to 7 of Investigation 1, "Are University Tuition Fees Rising Exponentially?" pp. 91 and 92

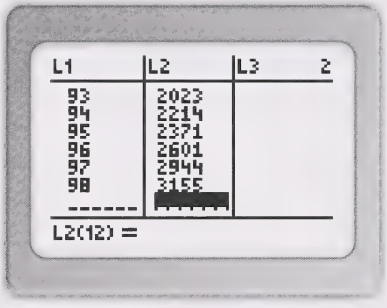
1. When you press STAT 1, you should see the following screen.



If there are entries in the lists, clear them as directed in question 1.

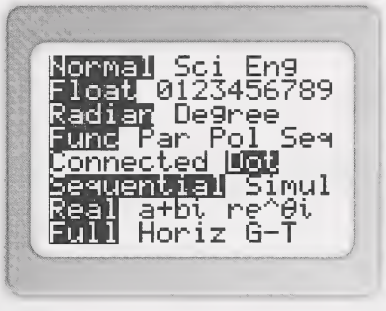
2. The simplest way to enter the data is to use the last two digits of the year school begins. For example, enter the 1988-89 school year as 88.

Your graphing calculator should look like the following once all the information has been entered.

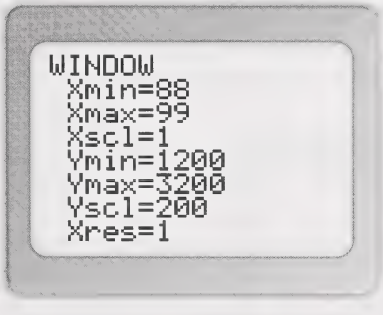


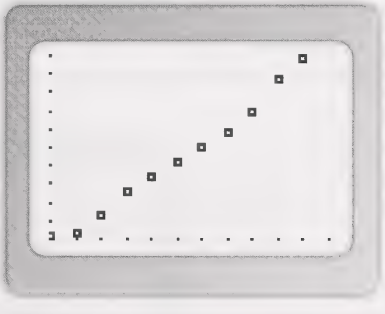
3. Turn Plot 1 on. Press 2nd [STAT PLOT] 1 ENTER.

Make sure your calculator is in Dot mode. Press MODE; use the arrow keys to select "Dot"; and press ENTER.



Press GRAPH to view the data using the window settings shown.

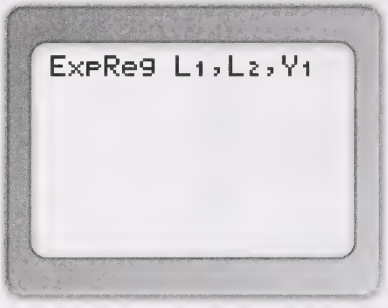


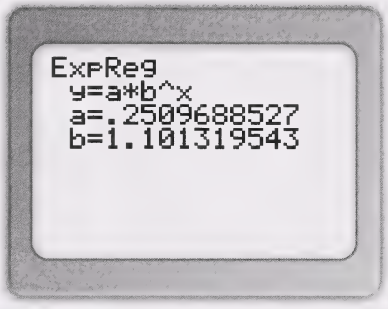


Yes, the data does appear to have an exponential curve pattern. However, scaling of the axes and the viewing window will affect the appearance of the pattern.

Section 1: Activity 1 (continued)

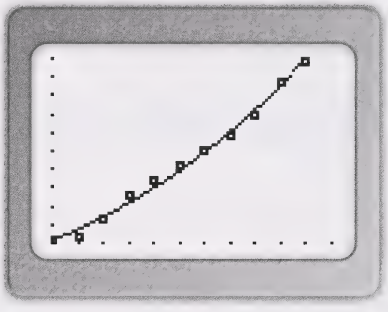
4. STAT [Select the CALC menu.] 0 (0:ExpReg) (2nd [L1] , 2nd [L2] , VARS [Select the Y-VARS menu.] 1 (1:Function...) 1 (1: Y_1) ENTER





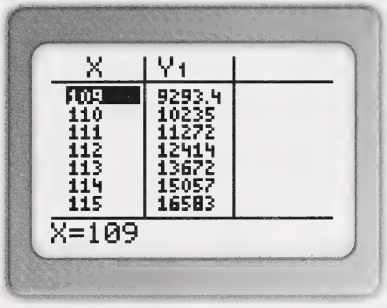
The equation is $y = 0.250\,968\,852\,7 \times 1.101\,319\,543^{x}$.

5. Press GRAPH to display the curve of best fit.



6. The exponential graph fits the data points. University tuition fees appear to be growing exponentially.

7. The year 2009–10 is year 109 for your exponential function. Press 2nd [TBLSET] and enter "109" for TblStart; then press 2nd TABLE.

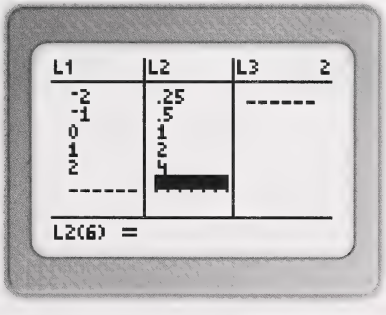


The average tuition for first-year students in 2009 will be approximately $9293. Some of the factors that might change tuition fees are government grants to universities or government policy on tuition levels.

6. Clear the lists you entered in question 5.

STAT 4 (4:ClrList) (2nd [L1] , 2nd [L2]) ENTER

Now, enter the data in columns L1 and L2.



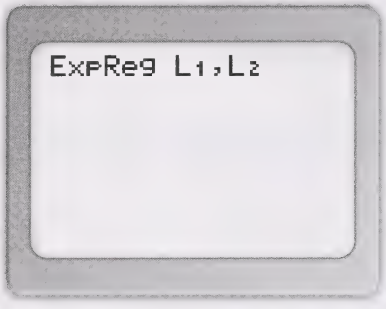
Section 1: Activity 1 (continued)

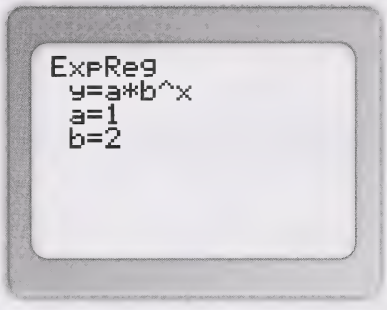
Press GRAPH to see the graph of these points. To view the data points, select the standard viewing window, ZOOM 6.

Note: Make sure you clear the equation in Y_1.

To obtain only the exponential equation, press the following:

STAT [Select the CALC menu.] 0 2nd [L1] , 2nd [L2] ENTER

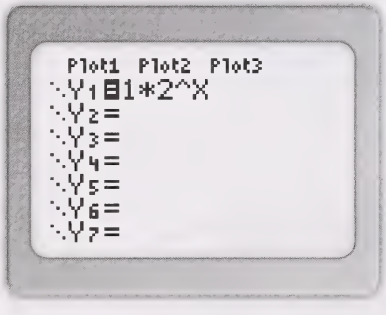


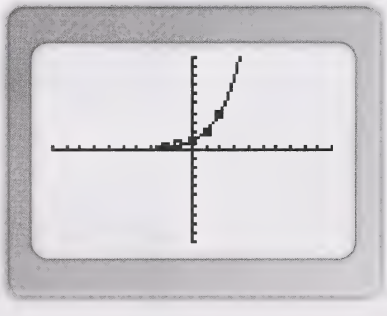


The regression formula is $y = 1(2^x)$ or $y = 2^x$.

Note: If, at this point, you want to graph this exponential equation, press the following:

Y= VARS 5 (5:Statistics) [Select the EQ menu.] 1 (1:RegEQ) GRAPH





7. Textbook questions 13 to 24 of "Mental Math: Working with Exponents," p. 71

13. $2^5 = 32$ **14.** $3^3 = 27$ **15.** $5^0 = 1$

16. $7^{-1} = \frac{1}{7}$ **17.** $4^{\frac{1}{2}} = 2$ **18.** $(-8)^{\frac{1}{3}} = -2$

19. $9^{-\frac{1}{2}} = \frac{1}{3}$ **20.** $8^{\frac{2}{3}} = 4$ **21.** $6^{-2} \times 6^2 = 1$

22. $(5^5)^{\frac{2}{5}} = 5^2$ or 25 **23.** $(-27)^{\frac{2}{3}} = 9$ **24.** $16^{-\frac{1}{4}} = \frac{1}{2}$

Section 1: Activity 2

1. a. Textbook question "Explore: Use Trial and Error," p. 86

Value of x	L.S. $= 8^{x-3}$	R.S. $= (2^{x-2})^2$
2	$8^{2-3} = 8^{-1}$ $= \frac{1}{8}$	$(2^{2-2})^2 = (2^0)^2$ $= 2^0$ $= 1$
3	$8^{3-3} = 8^0$ $= 1$	$(2^{3-2})^2 = (2^1)^2$ $= 2^2$ $= 4$
4	$8^{4-3} = 8^1$ $= 8$	$(2^{4-2})^2 = (2^2)^2$ $= 2^4$ $= 16$
5	$8^{5-3} = 8^2$ $= 64$	$(2^{5-2})^2 = (2^3)^2$ $= 2^6$ $= 64$

Both sides of the equation have the same value when $x = 5$.

Section 1: Activity 2 (continued)

b. Textbook questions 1 to 6 of "Inquire," p. 86

1. Answers may vary. The least value of x tested was 2, because it is the smallest integer for which $(2^{x-2})^2$ is also an integer.

2. Because $2^3 = 8$, the common base of 8 and 2 is 2.

$$\therefore\ 8^{x-3} = (2^{x-2})^2$$
$$(2^3)^{x-3} = (2^{x-2})^2$$
$$2^{3x-9} = 2^{2x-4}$$

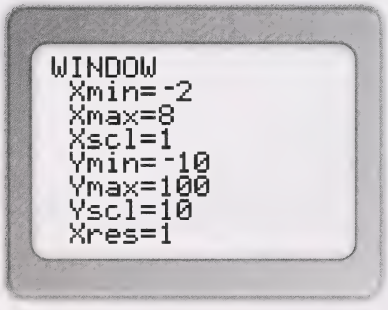
3. Because the bases are the same, you can now use a linear equation in x to solve the equation.

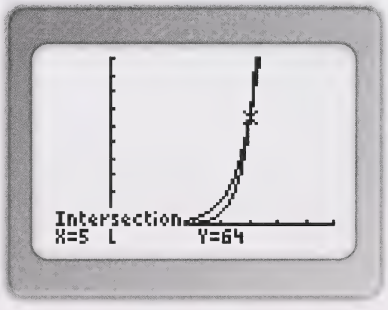
4.
$$2^{3x-9} = 2^{2x-4}$$
$$3x - 9 = 2x - 4$$
$$3x - 2x = -4 + 9$$
$$x = 5$$

This is the same value for x as you found using trial and error.

5. This question would be more difficult to do if the 8 were changed to 25 because the numbers 25 and 2 do not have a common base.

6. You could have solved the equation $8^{x-3} = (2^{x-2})^2$ by determining where the graphs of $y = 8^{x-3}$ and $y = (2^{x-2})^2$ intersect.





2. a. Textbook questions 1, 10, 13, and 16 of "Practice," p. 89

1.
$$27^x = 9^{2x-1}$$
$$(3^3)^x = (3^2)^{2x-1}$$
$$3^{3x} = 3^{4x-2}$$
$$3x = 4x - 2$$
$$-x = -2$$
$$x = 2$$

10.
$$3(5^{x+1}) = 15$$
$$5^{x+1} = 5$$
$$5^{x+1} = 5^1$$
$$x + 1 = 1$$
$$x = 0$$

13.
$$\sqrt[5]{256} \div \sqrt[6]{64} = 2^x$$
$$\sqrt[5]{2^8} \div \sqrt[6]{2^6} = 2^x$$
$$2^{\frac{8}{5}} \div 2^{\frac{6}{6}} = 2^x$$
$$2^{\frac{8}{5}} \div 2^1 = 2^x$$
$$2^{\frac{8}{5}-1} = 2^x$$
$$2^{\frac{8}{5}-\frac{5}{5}} = 2^x$$
$$2^{\frac{3}{5}} = 2^x$$
$$x = \frac{3}{5}$$

16.
$$8^{\frac{1}{4}} \times \left(\frac{1}{4}\right)^{\frac{x}{2}} = 16^{\frac{3}{4}}$$
$$(2^3)^{\frac{1}{4}} \left(\frac{1}{2^2}\right)^{\frac{x}{2}} = (2^4)^{\frac{3}{4}}$$
$$2^{\frac{3}{4}} (2^{-2})^{\frac{x}{2}} = 2^3$$
$$2^{\frac{3}{4}} (2^{-x}) = 2^3$$
$$2^{\frac{3}{4}-x} = 2^3$$
$$\frac{3}{4} - x = 3$$
$$-x = 2\frac{1}{4}$$
$$x = -2\frac{1}{4} \text{ or } -\frac{9}{4}$$

Section 1: Activity 2 (continued)

b. Textbook questions 24.a., 24.f., and 33 of "Applications and Problem Solving," p. 90

24. a.

$$\sqrt[5]{\frac{4^{3x-2}}{8^{x+1}}} = 16$$

$$\left(\sqrt[5]{\frac{4^{3x-2}}{8^{x+1}}}\right)^5 = 16^5$$

$$\frac{4^{3x-2}}{8^{x+1}} = 16^5$$

$$\frac{(2^2)^{3x-2}}{(2^3)^{x+1}} = (2^4)^5$$

$$\frac{2^{6x-4}}{2^{3x+3}} = 2^{20}$$

$$2^{(6x-4)-(3x+3)} = 2^{20}$$

$$2^{6x-4-3x-3} = 2^{20}$$

$$2^{3x-7} = 2^{20}$$

$$3x - 7 = 20$$

$$3x = 27$$

$$x = 9$$

f.

$$3^{x+1} + 3^x = 36$$

$$3^x \cdot 3^1 + 3^x = 36$$

$$3^x (3+1) = 36$$

$$3^x (4) = 36$$

$$3^x = 9$$

$$3^x = 3^2$$

$$x = 2$$

33. a.

$$(8x)^{-\frac{1}{2}} = \frac{1}{4}$$

$$\left[(8x)^{-\frac{1}{2}}\right]^{-2} = \left(\frac{1}{4}\right)^{-2}$$

$$(8x)^1 = 4^2$$

$$8x = 16$$

$$x = 2$$

Check

$(8x)^{-\frac{1}{2}}$	$\frac{1}{4}$
$= (8 \cdot 2)^{-\frac{1}{2}}$	
$= 16^{-\frac{1}{2}}$	
$= \frac{1}{16^{\frac{1}{2}}}$	
$= \frac{1}{\sqrt{16}}$	
$= \frac{1}{4}$	

LS = RS

b. This equation differs from other exponential equations in this section because the unknown, x, is part of the base rather than the exponent.

3. a. Textbook questions 17, 18, 20, and 23 of "Practice," p. 90

17. Let $A(t)$ represent the amount of strontium-90 after t years, and let A_0 represent the original amount.

Since the substance is decreasing by one-half every 25 years, the equation is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{25}}$.

If $\frac{1}{4}$ remains after t years, then $A(t) = \frac{1}{4}A_0$.

$$\therefore A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{25}}$$

$$\frac{1}{4}A_0 = A_0\left(\frac{1}{2}\right)^{\frac{t}{25}}$$

$$\frac{1}{4} = \left(\frac{1}{2}\right)^{\frac{t}{25}}$$

$$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{\frac{t}{25}}$$

$$2 = \frac{t}{25}$$

$$50 = t$$

$$t = 50$$

Therefore, 50 years have elapsed.

Section 1: Activity 2 (continued)

18. If 12.5% remains after t years, then $A(t) = 12.5\%$ of A_0.

$$A(t) = 12.5\% \text{ of } A_0$$
$$= \frac{12.5}{100} A_0$$
$$= \frac{1}{8} A_0$$

$$\therefore A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{25}}$$
$$\frac{1}{8} A_0 = A_0 \left(\frac{1}{2}\right)^{\frac{t}{25}}$$
$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{t}{25}}$$
$$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{t}{25}}$$
$$3 = \frac{t}{25}$$
$$t = 75$$

Therefore, 75 years have elapsed.

20. Let $A(t)$ represent the amount of plutonium-243 after t hours; let A_0 represent the original amount; and let h be the half-life.

Since the substance is decreasing by one-half every h hours, the equation is $A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{h}}$.

If $\frac{1}{64}$ of the original amount remains after 30 h, then $A(t) = \frac{1}{64} A_0$.

$$\therefore A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{h}}$$
$$\frac{1}{64} A_0 = A_0 \left(\frac{1}{2}\right)^{\frac{30}{h}}$$
$$\frac{1}{64} = \left(\frac{1}{2}\right)^{\frac{30}{h}}$$
$$\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^{\frac{30}{h}}$$
$$6 = \frac{30}{h}$$
$$6h = 30$$
$$h = 5$$

Therefore, the half-life of plutonium-243 is 5 h.

23. Let $A(t)$ represent the amount of radium-221 after t minutes; let A_0 represent the original amount; and let h represent the half-life.

Since the substance is decreasing by one-half every h minutes, the equation is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$.

If 6.25% of the original mass remains after 2 min, then $A(t) = 6.25\%$ of A_0.

$$\begin{aligned} A(t) &= 6.25\% \text{ of } A_0 \\ &= \frac{6.25}{100} A_0 \\ &= \frac{1}{16} A_0 \end{aligned}$$

$$\therefore A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$\frac{1}{16} A_0 = A_0\left(\frac{1}{2}\right)^{\frac{2}{h}}$$

$$\frac{1}{16} = \left(\frac{1}{2}\right)^{\frac{2}{h}}$$

$$\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{\frac{2}{h}}$$

$$4 = \frac{2}{h}$$

$$4h = 2$$

$$h = 0.5$$

The half-life of radium-221 is 0.5 min or 30 s.

Section 1: Activity 2 (continued)

b. Textbook questions 25 to 28 of "Applications and Problem Solving," p. 90

25. Let $N(t)$ represent the number of bacteria after t hours; let N_0 represent the initial bacteria count; and let d represent the doubling time.

$$\therefore N(t) = N_0 (2)^{\frac{t}{d}}$$

$$N\left(1\frac{1}{2}\right) = 80\ 000 \text{ and } N_0 = 1250$$

$$\therefore N(t) = N_0 (2)^{\frac{t}{d}}$$

$$N\left(1\frac{1}{2}\right) = 1250(2)^{\frac{1.5}{d}}$$

$$80\ 000 = 1250(2)^{\frac{1.5}{d}}$$

$$64 = 2^{\frac{1.5}{d}}$$

$$2^6 = 2^{\frac{1.5}{d}}$$

$$6 = \frac{1.5}{d}$$

$$6d = 1.5$$

$$d = \frac{1.5}{6}$$

$$= \frac{1}{4} \text{ or } 0.25$$

The doubling period for this bacterium is 0.25 h or 15 min.

26. a. Let $A(t)$ represent the amount of sodium-24 present after t hours.

There are 40 mg initially, and the substance is decreasing by one-half every 14.9 h.

$$\therefore A(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

After 48 hours, $t = 48$.

$$\therefore A(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

$$A(48) = 40\left(\frac{1}{2}\right)^{\frac{48}{14.9}}$$

$$\doteq 4.3$$

Approximately 4.3 mg of sodium-24 remain after 48 h.

b. Determine t if $A(t) = 2.5$ mg.

$$\therefore A(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

$$2.5 = 40\left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

$$\frac{2.5}{40} = \left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

$$\frac{1}{16} = \left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

$$\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{\frac{t}{14.9}}$$

$$4 = \frac{t}{14.9}$$

$$t = 4(14.9)$$

$$= 59.6$$

After 59.6 h, only 2.5 mg of sodium-24 remain.

Section 1: Activity 2 (continued)

27. a. After 10.6 years, $t = 10.6$.

$$\therefore A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$A(10.6) = A_0\left(\frac{1}{2}\right)^{\frac{10.6}{5.3}}$$

$$= A_0\left(\frac{1}{2}\right)^{2}$$

$$= \frac{1}{4}A_0$$

After 10.6 years, only $\frac{1}{4}$ of the original amount remains.

b. $A(t) = 12.5\%$ of A_0

$$= \frac{12.5}{100} \times A_0$$

$$= \frac{1}{8}A_0$$

$$\therefore A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$\frac{1}{8}A_0 = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$\left(\frac{1}{2}\right)^{3} = \left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

$$3 = \frac{t}{5.3}$$

$$t = 15.9$$

After 15.9 years, only 12.5% of the original amount remains.

28. a. Let $N(t)$ be the number of bacteria after t days, and let N_0 be the initial count.

Since the number of bacteria quadruples every 7 days, the equation is $N(t) = N_0 (4)^{\frac{t}{7}}$.

b. In 3 weeks, $t = 3 \times 7 = 21$.

$$\begin{aligned} \therefore N(t) &= N_0 (4)^{\frac{t}{7}} \\ N(21) &= N_0 (4)^{\frac{21}{7}} \\ &= N_0 (4)^3 \\ &= 64 N_0 \end{aligned}$$

In 3 weeks, there will be 64 times the original number of bacteria.

c. If the number doubles, $N(t) = 2N_0$.

$$\begin{aligned} \therefore N(t) &= N_0 (4)^{\frac{t}{7}} \\ 2N_0 &= N_0 (4)^{\frac{t}{7}} \\ 2 &= 4^{\frac{t}{7}} \\ 2 &= (2^2)^{\frac{t}{7}} \\ 2^1 &= 2^{\frac{2t}{7}} \\ 1 &= \frac{2t}{7} \\ 7 &= 2t \\ t &= 3.5 \end{aligned}$$

The number of bacteria doubles in 3.5 days.

Section 1: Activity 2 (continued)

d. $N(t) = 25\%$ of N_0

$$= \frac{1}{4} N_0$$

$$\therefore N(t) = N_0 (4)^{\frac{t}{7}}$$

$$\frac{1}{4} N_0 = N_0 (4)^{\frac{t}{7}}$$

$$\frac{1}{4} = 4^{\frac{t}{7}}$$

$$4^{-1} = 4^{\frac{t}{7}}$$

$$-1 = \frac{t}{7}$$

$$t = -7$$

There were only 25% of the current number of bacteria 7 days ago.

e. $N(t) = 8^{24}$ and $N_0 = 1$

$$\therefore N(t) = N_0 (4)^{\frac{t}{7}}$$

$$8^{24} = 1(4)^{\frac{t}{7}}$$

$$\left(2^3\right)^{24} = \left(2^2\right)^{\frac{t}{7}}$$

$$2^{72} = 2^{\frac{2t}{7}}$$

$$72 = \frac{2t}{7}$$

$$2t = 504$$

$$t = 252$$

It will take a single bacterium 252 days to grow to 8^{24} bacteria.

4. Textbook question 3 of "Applications and Problem Solving," p. 85

3. Let R be the number of correct answers, and let W be the number of wrong answers.

Since there are 33 items on the test, $R+W=33$.

Because a correct answer is worth 7 marks, an incorrect answer is worth -4 marks, and John scored 0 on his test,

$$7R-4W=0$$

Method 1: Guess and Check

Try $R=8$.

$\therefore R+W=33$
$8+W=33$
$W=25$

This guess is too low.

LS	RS
$7R-4W$	0
$=7(8)-4(25)$	
$=56-100$	
$=-44$	

LS $\neq$ RS

Try $R=11$.

$\therefore R+W=33$
$11+W=33$
$W=22$

This guess is too low.

LS	RS
$7R-4W$	0
$=7(11)-4(22)$	
$=77-88$	
$=-11$	

LS $\neq$ RS

Try $R=12$.

$\therefore R+W=33$
$12+W=33$
$W=21$

Therefore, John answered 12 questions correctly.

LS	RS
$7R-4W$	0
$=7(12)-4(21)$	
$=84-84$	
$=0$	

LS $=$ RS

Section 1: Activity 2 (continued)

Method 2: Algebraic Approach

$R+W=33$ ①
$7R-4W=0$ ②

$4R+4W=132$ ③ : $4\times$ ①
$7R-4W=0$ ②
$11R=132$ ③ + ②
$R=12$

Therefore, John answered 12 questions correctly.

Section 1: Follow-up Activities

Extra Help

1. Textbook questions 2.a. and 3.a. of "Chapter Check," p. 128

2. a. Work from $A(t)=P(1+i)^n$.

The trout population, $P(t)$, after t years corresponds to the amount, $A(t)$.

The current trout population, 200, corresponds to the original principal, P.

If the population doubles, it increases by 100%. This increase corresponds to the interest rate per period, i. Therefore, $i=100\%$ or 1.00.

The trout population doubles every 5 years. The number of doubling periods in t years is $\frac{t}{5}$. Therefore, $n=\frac{t}{5}$.

Substitute this information into the interest formula.

$$A(t)=P(1+i)^n$$
$$P(t)=200(1+1.00)^{\frac{t}{5}}$$
$$=200(2)^{\frac{t}{5}}$$

The equation that gives the trout population in t years is $P(t)=200(2)^{\frac{t}{5}}$.

3. a. Work from $A(t) = P(1+i)^n$.

The value, $V(t)$, of the digger after t years corresponds to the amount, $A(t)$.

The purchase price, \$140 000, corresponds to the original principal, P.

The value of the digger decreases by 12% per year. This decrease corresponds to the interest rate per period, i. Therefore, $i = -12\%$ or -0.12.

The value of the digger depreciates annually. Therefore, $n = t$.

Substitute this information into the interest formula.

$$A(t) = P(1+i)^n$$
$$V(t) = 140\,000(1-0.12)^t$$
$$= 140\,000(0.88)^t$$

The equation that gives the value of the digger in t years is $V(t) = 140\,000(0.88)^t$.

2. Work from $A(t) = P(1+i)^n$.

The value, $V(t)$, of the card after t years corresponds to the amount, $A(t)$.

The current value, \$150, corresponds to the principal, P.

The value of the card increases by 20% per year. This increase corresponds to the interest rate per period, i. Therefore, $i = 20\%$ or 0.20.

The value of the card increases annually. Therefore, $n = t$.

Substitute this information into the interest formula.

$$A(t) = P(1+i)^n$$
$$V(t) = 150(1+0.20)^t$$
$$= 150(1.2)^t$$

The equation that gives the card's value in t years is $V(t) = 150(1.2)^t$.

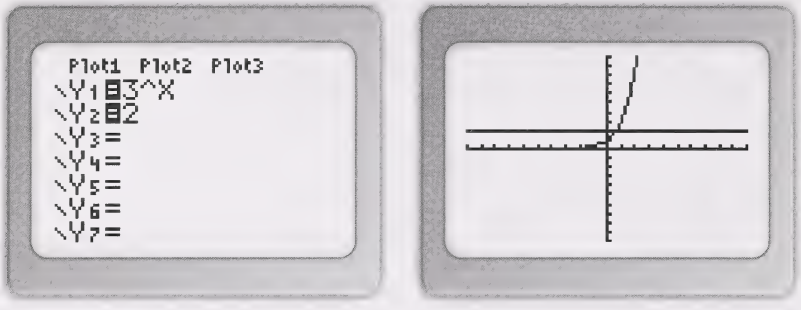
Enrichment

1. $y = 3^x$ (1)

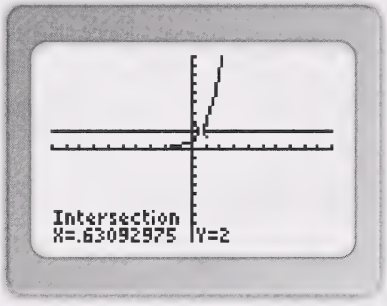
$y = 2$ (2)

Graph the system of equations on your graphing calculator using the standard window settings.

Y= 3 ^ X,T,Θ,n ENTER 2 GRAPH



Now, use the Intersect feature from the CALCULATE menu to determine where the two graphs intersect.



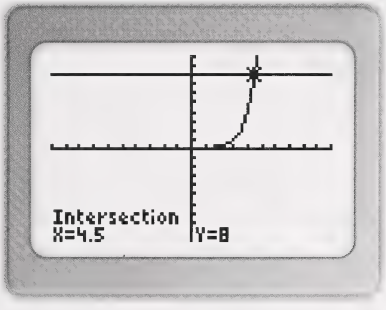
Therefore, $x \doteq 0.630\,929\,75$.

2. $y = 4^{x-3}$ (1)

$y = 8$ (2)

Graph the system of equations on your graphing calculator.

Now, use the Intersect feature from the CALCULATE menu to determine where the two graphs intersect.



Therefore, $x = 4.5$.

Section 2: Activity 1

1. a. Textbook question "Explore: Rewrite in Another Form," p. 94

Logarithmic Form	Exponential Form	Value of y
$y = \log_2 16$	$16 = 2^y$	4
$y = \log_2 64$	$64 = 2^y$	6
$y = \log_2 \frac{1}{2}$	$\frac{1}{2} = 2^y$	-1
$y = \log_{10} 100$	$100 = 10^y$	2
$y = \log_{10} 1000$	$1000 = 10^y$	3
$y = \log_{10} 0.1$	$0.1 = 10^y$	-1
$y = \log_3 81$	$81 = 3^y$	4
$y = \log_5 125$	$125 = 5^y$	3

b. Textbook questions 1 to 6 of "Inquire," p. 94

1. $\log_2 n = 5$

$n = 2^5$

$= 32$

2. $\log_2 16 = k$

$16 = 2^k$

$2^4 = 2^k$

$k = 4$

3. $\log_b 49 = 2$

$49 = b^2$

$b = 7$

4. The range of $y = b^n$ is $y > 0$. Because $y = \log_b n$ is the inverse of the previous function, the domain of $\log_b n$ is $n > 0$. **Remember:** For inverse functions, domain and range are interchanged.

5. a. Yes, $\log_b n$ could be a negative number. For example, $\log_2 \left(\frac{1}{2}\right) = y$ written in exponential form is $\frac{1}{2} = 2^y$. Therefore, $y = -1$.

b. Yes, $\log_b n$ could be zero.

If $\log_b n = 0$, then

$$n = b^0$$
$$= 1$$

$$\therefore \log_b 1 = 0$$

6. Because $3^3 = 27$ and $3^4 = 81$ and $3^3 < 59 < 3^4$, then $3 < \log_3 59 < 4$.

2. a. Textbook questions 2, 4, 5, 6, 10, 11, 13, 15, 19, 21, 23, 29, 32, 33, 39, and 40 of "Practice," pp. 98 and 99

2. $6^3 = 216$

$\therefore 3 = \log_6 216$

4. $8^0 = 1$

$\therefore 0 = \log_8 1$

5. $49^{\frac{1}{2}} = 7$

$\therefore \frac{1}{2} = \log_{49} 7$

6. $5^{-2} = \frac{1}{25}$

$\therefore -2 = \log_5 \left(\frac{1}{25}\right)$

10. $a^b = c$

$\therefore b = \log_a c$

11. $\log_5 5 = 1$

$\therefore 5^1 = 5$

13. $\log_3 729 = 6$

$\therefore 3^6 = 729$

15. $\log_{10} 1 = 0$

$\therefore 10^0 = 1$

19. $\log 0.1 = -1$

$\therefore 10^{-1} = 0.1$

Note:
log 0.1 = −1 means
$\log_{10}$ 0.1 = −1

Section 2: Activity 1 (continued)

21. Let $x = \log_2 32$.

$\therefore\ 2^x = 32$ — Write in exponential form.

$2^x = 2^5$

$x = 5$

$\therefore\ \log_2 32 = 5$

23. Let $x = \log 1000$.

$\therefore\ 10^x = 1000$ — Write in exponential form

$10^x = 10^3$

$x = 3$

$\therefore\ \log 1000 = 3$

29. Let $x = \log_2 0.25$.

$\therefore\ 2^x = 0.25$ — Write in exponential form

$2^x = \frac{1}{4}$

$2^x = \frac{1}{2^2}$

$2^x = 2^{-2}$

$x = -2$

$\therefore\ \log_2 0.25 = -2$

32. Let $x = \log_2 \left(8\sqrt{2}\right)$.

$\therefore\ 2^x = 8\sqrt{2}$ — Write in exponential form.

$2^x = 2^3 \cdot 2^{\frac{1}{2}}$

$2^x = 2^{3\frac{1}{2}}$

$x = 3\frac{1}{2}$

$\therefore\ \log_2 \left(8\sqrt{2}\right) = 3\frac{1}{2}$

33. $\log_3 81 = x$

$3^x = 81$

$3^x = 3^4$

$x = 4$

39. $\log_x 16 = \frac{4}{3}$

$x^{\frac{4}{3}} = 16$

$\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = (16)^{\frac{3}{4}}$

$x^1 = \left(\sqrt[4]{16}\right)^3$

$x = 2^3$

$= 8$

40. $\log_9 3\sqrt{3} = x$

$$9^x = 3\sqrt{3}$$
$$(3^2)^x = 3^1 \times 3^{\frac{1}{2}}$$
$$3^{2x} = 3^{1\frac{1}{2}}$$
$$2x = 1\frac{1}{2}$$
$$2x = \frac{3}{2}$$
$$x = \frac{3}{2} \times \frac{1}{2}$$
$$= \frac{3}{4}$$

b. Textbook questions 41 to 43 of "Applications and Problem Solving," p. 99

41. a. Prove $\log_b 1 = 0$, where $b > 0$ and $b \neq 1$.

Method 1: Using Logarithms

LS	RS
$\log_b 1$	0
$= \log_b b^0$	
$= 0$	

LS = RS

$\therefore \log_b 1 = 0$

Method 2: Using an Exponential Equation

Let $x = \log_b 1$.

$\therefore b^x = 1$

$b^x = b^0$

$x = 0$

$\therefore \log_b 1 = 0$

b. Prove $\log_b b = 1$, where $b > 0$ and $b \neq 1$.

Method 1: Using Logarithms

LS	RS
$\log_b b$	1
$= \log_b b^1$	
$= 1$	

LS = RS

$\therefore \log_b b = 1$

Method 2: Using an Exponential Equation

Let $x = \log_b b$.

$\therefore b^x = b$

$b^x = b^1$

$x = 1$

$\therefore \log_b b = 1$

Section 2: Activity 1 (continued)

c. Prove $b^{\log_b x} = x$, where $b > 0$ and $b \neq 1$.

Let $y = b^{\log_b x}$.

$\therefore \log_b x = \log_b y$

$x = y$

$x = b^{\log_b x}$

$\therefore b^{\log_b x} = x$

42. Suppose $\log_4 x = 2$.

$\therefore x = 4^2$

$= 16$

If $\log_4 x < 2$, then $0 < x < 16$.

43. Because $3^4 = 81$ and $3^5 = 243$ and $81 < 100 < 243$, then $4 < \log_3 100 < 5$. Therefore, if $x = \log_3 100$, then $4 < x < 5$.

3.

Graph	$y = 3^x$	$y = 0.2^x$	$y = \log_3 x$	$y = \log_{0.2} x$
Domain	Reals	Reals	$x > 0$	$x > 0$
Range	$y > 0$	$y > 0$	Reals	Reals
x-intercept	none	none	1	1
y-intercept	1	1	none	none
Horizontal Asymptote	$y = 0$	$y = 0$	none	none
Vertical Asymptote	none	none	$x = 0$	$x = 0$

4. For bases greater than 1, the graph of $y = \log_b x$ becomes less steep as b increases. If b is replaced with $\frac{1}{b}$, the resulting graph, $y = \log_{\frac{1}{b}} x$, is the reflection in the x-axis of $y = \log_b x$, where $b > 1$.

In all cases, the x-intercept is 1, there is no y-intercept, the domain is $x > 0$, the range is the set of reals, and the vertical asymptote is $x = 0$.

5. Textbook question 44 of "Applications and Problem Solving," p. 99

44. a. Make a small table of values for each function. Use the exponential form of each to help select convenient choices for x.

$y = \log_2 x$

$x = 2^y$

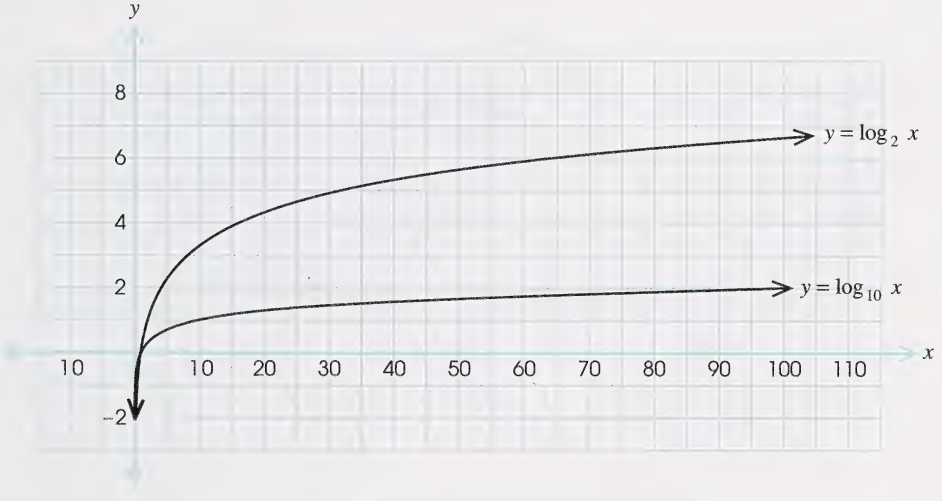
x	0.25	0.5	1	2	4	8	16	32	64
y	−2	−1	0	1	2	3	4	5	6

$y = \log_{10} x$

$x = 10^y$

x	0.01	0.1	1	10	100
y	−2	−1	0	1	2

Use these tables to graph $y = \log_2 x$ and $y = \log_{10} x$ in the domain $0 < x \leq 100$.



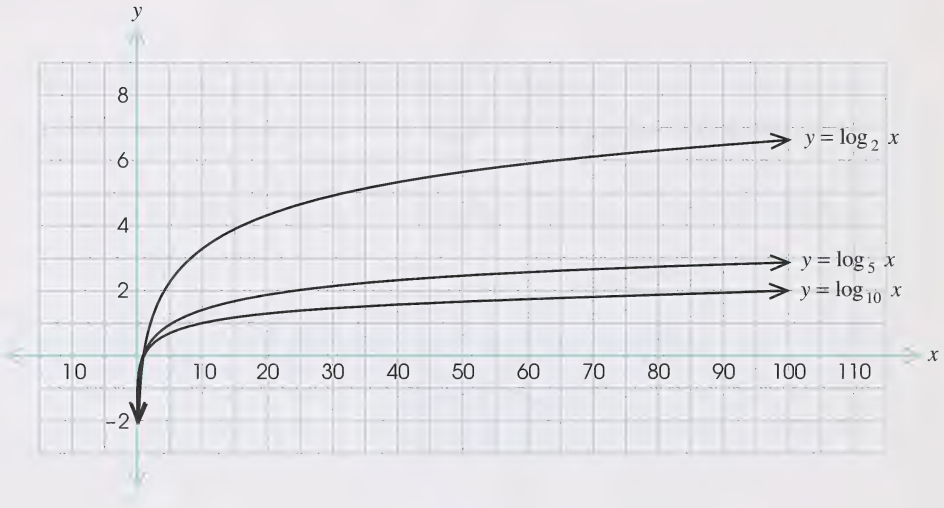
Section 2: Activity 1 (continued)

b. The value of 5 is between 2 and 10; therefore, the graph of $y = \log_5 x$ will be between the graphs of $y = \log_2 x$ and $y = \log_{10} x$.

For any chosen y-value, the x-value of $y = \log_5 x$ can be found by dividing the corresponding x-value of $y = \log_{10} x$ by that of $y = \log_2 x$. For example, when $y = 2$, $x = \frac{100}{4} = 25$.

Use this procedure to set up a table of values; then plot and join the points.

x	$\frac{0.01}{0.25} = \frac{1}{25}$	$\frac{0.1}{0.5} = \frac{1}{5}$	$\frac{1}{1} = 1$	$\frac{10}{2} = 5$	$\frac{100}{4} = 25$	$\frac{1000}{8} = 125$
y	-2	-1	0	1	2	3



6. Textbook questions 46, 49, 50.a., and 55 of "Applications and Problem Solving," pp. 99 and 100

46. a.
$$\frac{I_{\text{Japan}}}{I_{\text{Alaska}}} = \frac{10^{8.9}}{10^{8.5}}$$
$$= 10^{8.9-8.5}$$
$$= 10^{0.4}$$
$$\doteq 2.5$$

The earthquake in Japan was about 2.5 times more intense than the earthquake in Alaska.

b.
$$\frac{I_{\text{Japan}}}{I_{\text{Guatemala}}} = \frac{10^{8.9}}{10^{7.5}}$$
$$= 10^{8.9-7.5}$$
$$= 10^{1.4}$$
$$\doteq 25$$

The earthquake in Japan was about 25 times more intense than the earthquake in Guatemala.

c.
$$\frac{I_{\text{Alaska}}}{I_{\text{Guatemala}}} = \frac{10^{8.5}}{10^{7.5}}$$
$$= 10^{8.5-7.5}$$
$$= 10^{1}$$
$$= 10$$

The earthquake in Alaska was 10 times more intense than the earthquake in Guatemala.

d.
$$\frac{I_{\text{Pacific}}}{I_{\text{Guatemala}}} = 0.5$$
$$\frac{I_{\text{Pacific}}}{10^{7.5}} = 0.5$$
$$I_{\text{Pacific}} = 0.5 \times 10^{7.5}$$
$$\doteq 158\,113\,88$$
$$\doteq 10^{7.2}$$ Convert to a power of 10.

The intensity of the Pacific earthquake was about $10^{7.2}$; therefore, the magnitude of this earthquake is about 7.2 on the Richter scale.

Section 2: Activity 1 (continued)

49. Use the formula shade $\# = \frac{7(-\log_{10} T)}{3} + 1$, where T is the fraction of visible light transmitted through the glass.

For #10 welding glasses,

$$10 = \frac{7(-\log_{10} T)}{3} + 1$$
$$9 = \frac{7(-\log_{10} T)}{3}$$
$$27 = 7(-\log_{10} T)$$
$$\frac{27}{7} = -\log_{10} T$$
$$-\frac{27}{7} = \log_{10} T$$
$$T = 10^{-\frac{27}{7}}$$

For #2 glasses,

$$2 = \frac{7(-\log_{10} T)}{3} + 1$$
$$1 = \frac{7(-\log_{10} T)}{3}$$
$$3 = 7(-\log_{10} T)$$
$$\frac{3}{7} = -\log_{10} T$$
$$-\frac{3}{7} = \log_{10} T$$
$$T = 10^{-\frac{3}{7}}$$

Compare the T-values.

$$\frac{T_{\#2}}{T_{\#10}} = \frac{10^{-\frac{3}{7}}}{10^{-\frac{27}{7}}}$$
$$= 10^{-\frac{3}{7}+\frac{27}{7}}$$
$$= 10^{\frac{24}{7}}$$
$$\doteq 2683$$

Approximately 2683 times more light is transmitted through #2 welding glasses than through #10 welding glasses.

50. a. $R = 0.67 \log(0.37E) + 1.46$

$8.9 = 0.67 \log(0.37E) + 1.46$

$7.44 = 0.67 \log(0.37E)$

$\frac{7.44}{0.67} = \log(0.37E)$

$0.37E = 10^{\frac{7.44}{0.67}}$

$E = \frac{10^{\frac{7.44}{0.67}}}{0.37}$

$\doteq 3.44 \times 10^{11}$

Approximately 3.44×10^{11} kilowatt-hours of energy was released by the 1933 Japan earthquake.

55. $y = \log\left(\frac{x}{x-1}\right) - x$

Because logarithms of negative numbers are undefined, $\frac{x}{x-1} > 0$.

$\therefore\ x > 0$ and $x - 1 > 0$ or $x < 0$ and $x - 1 < 0$.

Case 1: $x > 0$ and $x - 1 > 0$

$x > 1$

$\therefore\ x > 1$

Case 2: $x < 0$ and $x - 1 < 0$

$x < 1$

$\therefore\ x < 0$

The domain of $y = \log\left(\frac{x}{x-1}\right) - x$ is $x < 0$ or $x > 1$.

Section 2: Activity 1 (continued)

7. Textbook question "WORD POWER," p. 100

Answers may vary. A sample answer is given.

MINE
MINT
MIST
MOST
MOAT
COAT
COAL

Section 2: Activity 2

1. a. Textbook questions a. to c. of "Explore: Use a Model," p. 102

a.

A	1	2	4	8	16	32	64	128
B	1	2	4	8	16	32		

$\therefore\ 4 \times 16 = 64$

b.

A	1	2	4	8	16	32	64	128	256	512
B	1	2	4	8	16	32				

$\therefore\ 16 \times 16 = 256$

c.

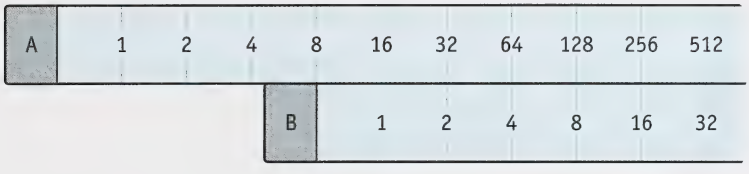
A	1	2	4	8	16	32	64	128	256
B	1	2	4	8					

$\therefore\ 32 \times 4 = 128$

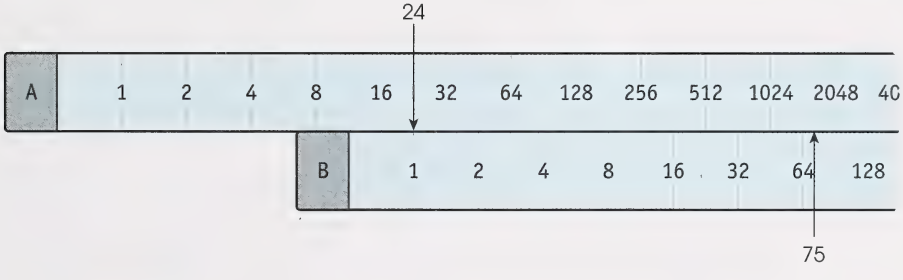
b. Textbook questions 1 to 5 of "Inquire," p. 102

1. This slide rule is a base-2 slide rule. Powers of 2 are equally spaced on the scales. When ruler B is positioned under ruler A to find the product of 8×32 or $2^3 \times 2^5$, for example, count off 5 divisions along ruler B. These 5 divisions are added to your starting point on ruler A. You end at 2^8 or 256 on ruler A. Here, you are using the product of powers property, $b^x \times b^y = b^{x+y}$.

2. To find the quotient of $128 \div 8$, slide ruler B below ruler A until point 8 on its scale aligns with the 128 on the scale of ruler A. Now, locate 1 on ruler B and read the number above it. Therefore, $128 \div 8 = 16$.



3. To find the value of 24×75, slide ruler B below ruler A until 1 on its scale aligns with the approximate location of 24 on the scale of ruler A. Now, locate the approximate location of 75 on ruler B and estimate the number above it.



The result is between 1024 and 2048 and is closer to 2048 than 1024.

The actual value is 1800.

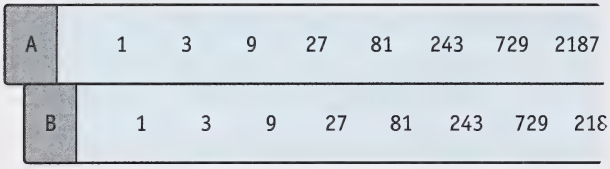
This process is not very accurate because you have to estimate the locations of 24, 75, and the answer.

Section 2: Activity 2 (continued)

4. If $2^0 = 1$ and $2^1 = 2$ and if x is the midpoint between 1 and 2, then

$$\begin{aligned} x &= 2^{0.5} \\ &= 2^{\frac{1}{2}} \\ &= \sqrt{2} \end{aligned}$$

5. To make a simple base-3 slide rule, use two strips of heavy paper at least 15 cm long. Mark 1-cm intervals and label the intervals as shown. Continue the scales to the right as far as you can by extending the pattern. Use the base-3 slide rule in a similar way as your base-2 slide rule.



2. a. Textbook questions 1, 3, 5, 7, 19, and 21 of "Practice," p. 106

1. $\log_{10} 8 + \log_{10} 1.25 = \log_{10} (8 \times 1.25)$
$= \log_{10} 10$
$= \log_{10} 10^1$
$= 1$

3. $\log_3 108 - \log_3 4 = \log_3 \left(\frac{108}{4}\right)$
$= \log_3 27$
$= \log_3 3^3$
$= 3$

5. $\log_6 4 + \log_6 9 = \log_6 (4 \times 9)$
$= \log_6 36$
$= \log_6 6^2$
$= 2$

7. $\log_3 8 - \log_3 24 = \log_3 \left(\frac{8}{24}\right)$
$= \log_3 \left(\frac{1}{3}\right)$
$= \log_3 3^{-1}$
$= -1$

19. $\log 170 = \log(17 \times 10)$
$= \log 17 + \log 10$
$= k + 1$

21. $\log 1.7 = \log\left(\frac{17}{10}\right)$
$= \log 17 - \log 10$
$= k - 1$

b. Textbook questions 43, 44.a., and 45 of "Applications and Problem Solving," p. 106

43.

LS	RS
$\log\left(\frac{1}{x}\right)$	$-\log x$
$= \log 1 - \log x$	
$= \log 10^0 - \log x$	
$= 0 - \log x$	
$= -\log x$	

LS = RS

$\therefore \log\left(\frac{1}{x}\right) = -\log x$

44. a. $\log y = \log x + \log 3$

$\log y = \log 3x$

$y = 3x$, where $x > 0$

The domain is $x > 0$.

45. Let $\log_b x = M$ and $\log_b y = N$.

Therefore, by definition, $x = b^M$ and $y = b^N$

$\therefore \frac{x}{y} = \frac{b^M}{b^N}$

$= b^{M-N}$

Express $\frac{x}{y} = b^{M-N}$ in logarithmic form.

$\log_b \frac{x}{y} = M - N$

$= \log_b x - \log_b y$

Section 2: Activity 2 (continued)

3. a. Textbook questions 11, 13, 16, 17, 22, and 26 of "Practice," p. 106

11. $\log_3 9^{20} = 20 \log_3 9$

$= 20 \log_3 3^2$

$= 20(2)$

$= 40$

13. $\log_{10} \sqrt{0.1} = \log_{10} 0.1^{\frac{1}{2}}$

$= \frac{1}{2} \log_{10} 0.1$

$= \frac{1}{2} \log_{10} 10^{-1}$

$= \frac{1}{2}(-1)$

$= -\frac{1}{2}$

16. $\log_7 49^{2.5} = 2.5 \log_7 49$

$= 2.5 \log_7 7^2$

$= 2.5(2)$

$= 5$

17. $\log_8 2^{\frac{3}{2}} = \log_8 (2^3)^{\frac{1}{2}}$

$= \frac{1}{2} \log_8 2^3$

$= \frac{1}{2} \log_8 8$

$= \frac{1}{2}(1)$

$= \frac{1}{2}$

22. $\log \sqrt{17} = \log 17^{\frac{1}{2}}$

$= \frac{1}{2} \log 17$

$= \frac{1}{2}(k)$

$= \frac{k}{2}$

26. $\log_3 2 = \log_3 \sqrt{4}$

$= \log_3 4^{\frac{1}{2}}$

$= \frac{1}{2} \log_3 4$

$= \frac{1}{2}(x)$

$= \frac{x}{2}$

b. Textbook questions 44.b. and 46 of "Applications and Problem Solving," p. 106

44. b. $\log y = 3 \log x$

$\log y = \log x^3$

$y = x^3$, where $x > 0$

The domain is $x > 0$.

46. a.

LS	RS
$\log_b x^{\frac{1}{n}}$	$\frac{\log_b x}{n}$
$= \frac{1}{n} \log_b x$	
$= \frac{\log_b x}{n}$	

LS = RS

b.

LS	RS
$\log_b \sqrt[n]{x^m}$	$\frac{m \log_b x}{n}$
$= \log_b (x^m)^{\frac{1}{n}}$	
$= \frac{1}{n} \log_b x^m$	
$= \frac{\log_b x^m}{n}$	
$= \frac{m \log_b x}{n}$	

LS = RS

4. a. Textbook questions 28, 31, 33, 36, and 42 of "Practice," p. 106

28. $\log_3 \sqrt[5]{4^7} = \frac{\log_3 4^7}{5}$

$= \frac{7 \log_3 4}{5}$

$= \frac{7x}{5}$

31. $\log_3 \frac{x^4}{27} = \log_3 x^4 - \log_3 27$

$= 4 \log_3 x - \log_3 3^3$

$= 4(8) - 3$

$= 29$

Section 2: Activity 2 (continued)

33. $2\log_3 12 - 2\log_3 4 = \log_3 12^2 - \log_3 4^2$

$= \log_3 \dfrac{12^2}{4^2}$

$= \log_3 3^2$

$= 2$

36. $\dfrac{1}{2}\log_3 144 - \log_3 4 + 2\log_3 3 = \log_3 \sqrt{144} - \log_3 4 + \log_3 3^2$

$= \log_3 12 - \log_3 4 + \log_3 9$

$= \log_3 \dfrac{12(9)}{4}$

$= \log_3 27$

$= \log_3 3^3$

$= 3$

42. $\log_7 \sqrt[3]{x} - \log_7 y^3 + 2\log_7 y = \log_7 \sqrt[3]{x} - \log_7 y^3 + \log_7 y^2$

$= \log_7 \dfrac{\sqrt[3]{x}y^2}{y^3}$

$= \log \dfrac{\sqrt[3]{x}}{y}$

b. Textbook questions 44.c. and 52 of "Applications and Problem Solving," pp. 106 and 107

44. c. $\log y = \dfrac{1}{2}(\log x + \log 5)$

$\log y = \dfrac{1}{2}\log 5x$

$\log y = \log \sqrt{5x}$

$y = \sqrt{5x}$, where $x > 0$

The domain is $x > 0$.

52. The functions $y = \log x^2$ and $y = 2\log x$ are not the same because their domains differ. The domain of $y = \log x^2$ is $x \neq 0$, whereas the domain of $y = 2\log x$ is $x > 0$.

5. Textbook questions 47.b., 47.c., 48, and 50 of "Applications and Problem Solving," p. 107

47. b.

$$\begin{aligned} \text{pH} &= -\log_{10}[\text{H}^+] \\ 5 &= -\log_{10}[\text{H}^+] \\ \log_{10}[\text{H}^+] &= -5 \\ \log_{10}[\text{H}^+] &= \log_{10} 10^{-5} \\ \text{H}^+ &= 10^{-5} \end{aligned}$$

The hydrogen ion concentration of black coffee is 10^{-5} moles per litre.

c.

$$\begin{aligned} \text{pH} &= -\log_{10}[\text{H}^+] \\ 7.8 &= -\log_{10}[\text{H}^+] \\ \log_{10}[\text{H}^+] &= -7.8 \\ \log_{10}[\text{H}^+] &= \log_{10} 10^{-7.8} \\ \text{H}^+ &= 10^{-7.8} \end{aligned}$$

The hydrogen ion concentration of eggs is $10^{-7.8}$ moles per litre.

48. a.

$$\begin{aligned} \text{pH} &= 6.1 + \log_{10}\frac{B}{C} \\ &= 6.1 + \log_{10} B - \log_{10} C \end{aligned}$$

b. Method 1: Using the Original Formula

$$\begin{aligned} \text{pH} &= 6.1 + \log_{10}\frac{B}{C} \\ &= 6.1 + \log_{10}\frac{25}{2} \\ &= 6.1 + \log_{10} 12.5 \\ &\doteq 7.2 \end{aligned}$$

Method 2: Using the Formula in 48.a.

$$\begin{aligned} \text{pH} &= 6.1 + \log_{10} B - \log_{10} C \\ &= 6.1 + \log_{10} 25 - \log_{10} 2 \\ &\doteq 7.2 \end{aligned}$$

Section 2: Activity 2 (continued)

50. $\log T = \dfrac{\log k + 3 \log r}{2}$

$\log T = \dfrac{\log k + \log r^3}{2}$

$\log T = \dfrac{\log (kr^3)}{2}$

$\log T = \log \sqrt{kr^3}$

$T = \sqrt{kr^3}$ or $T^2 = kr^3$

The equation in exponential form is $T = \sqrt{kr^3}$ or $T^2 = kr^3$.

6. Textbook question "NUMBER POWER," p. 107

Answers may vary. A sample answer is given.

6 1 10 8
5 9 2
4 7
3

Section 2: Activity 3

1. a. Textbook question "Explore: Solve an Equation," p. 108

$3^{x+2} = 5^{2x-3}$

Using trial and error, $x \doteq 3.3$.

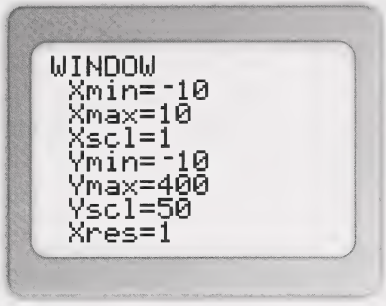
Check

LS	RS
3^{x+2}	5^{2x-3}
$\doteq 3^{3.3+2}$	$\doteq 5^{2(3.3)-3}$
$\doteq 3^{5.3}$	$\doteq 5^{3.6}$
$\doteq 338$	$\doteq 328$

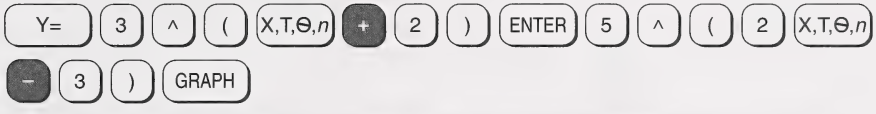
LS $\doteq$ RS

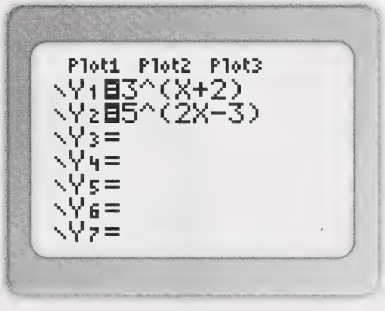
Now, use your graphing calculator to obtain another estimate.

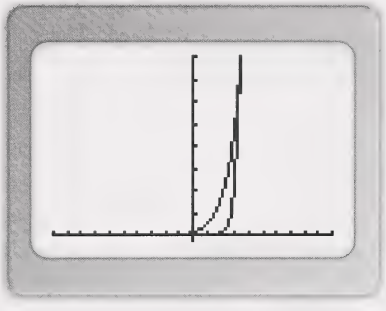
Use the following window settings.



Graph $y = 3^{x+2}$ and $y = 5^{2x-3}$.







Use the Intersect feature from the CALCULATE menu to determine where the two graphs intersect.

Section 2: Activity 3 (continued)

b. Textbook questions 1 to 5 of "Inquire," p. 108

1. Trial and error has several disadvantages. It is time consuming, inefficient, and difficult to narrow the possibilities. Also, the answers are approximate, which may cause additional solutions to be overlooked.

2. The domain of each exponential function is the set of real numbers. The range of each function is $y > 0$.

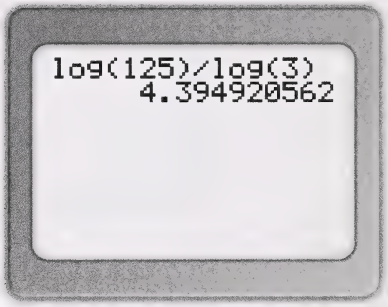
3. Trying to solve the equation using a graph drawn manually also has several disadvantages. It is difficult to decide on an appropriate scale unless you know the answer beforehand. Also, the answers are only approximations, which may cause additional solutions to be overlooked.

4. There are disadvantages when using a graphing calculator. Unless you know what the answer will be, or at least are able to estimate the answer, you will have to adjust the viewing window until you can view the point of intersection. Again, the solution is approximate and you may not see all possible solutions in the window you selected.

5. You cannot solve this exponential equation by equating exponents because the bases differ.

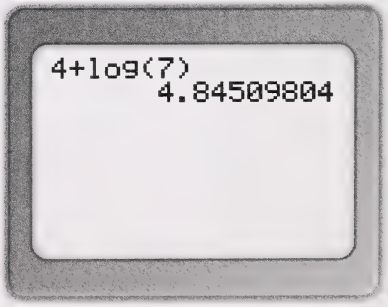
2. Textbook questions 1, 2, 4, 9, and 13 of "Practice," p. 113

1.
$$3^x = 125$$
$$\log 3^x = \log 125$$
$$x \log 3 = \log 125$$
$$x = \frac{\log 125}{\log 3}$$
$$\doteq 4.39$$



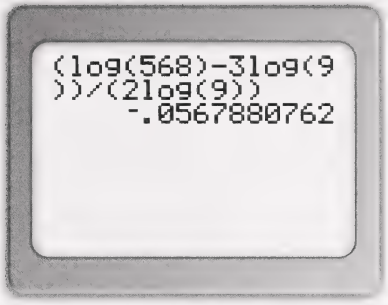
2.
$$10^{x-4} = 7$$
$$\log 10^{x-4} = \log 7$$
$$x - 4 = \log 7$$
$$x = 4 + \log 7$$
$$\doteq 4.85$$

$\log 10^m = m$



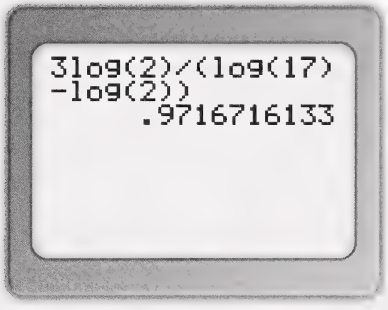
4.

$$9^{2x+3} = 568$$
$$\log 9^{2x+3} = \log 568$$
$$(2x+3)\log 9 = \log 568$$
$$2x \log 9 + 3 \log 9 = \log 568$$
$$2x \log 9 = \log 568 - 3 \log 9$$
$$x = \frac{\log 568 - 3 \log 9}{2 \log 9}$$
$$\doteq -0.06$$



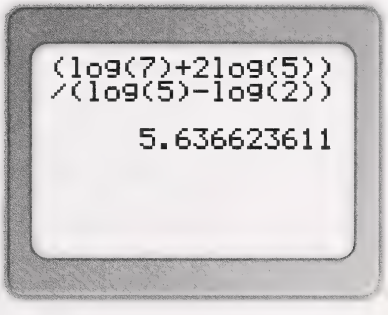
9.

$$2^{x+3} = 17^{x}$$
$$\log 2^{x+3} = \log 17^{x}$$
$$(x+3)\log 2 = x \log 17$$
$$x \log 2 + 3 \log 2 = x \log 17$$
$$3 \log 2 = x \log 17 - x \log 2$$
$$3 \log 2 = x(\log 17 - \log 2)$$
$$x(\log 17 - \log 2) = 3 \log 2$$
$$x = \frac{3 \log 2}{\log 17 - \log 2}$$
$$\doteq 0.972$$



13.

$$7 \times 2^{x} = 5^{x-2}$$
$$\log(7 \times 2^{x}) = \log(5^{x-2})$$
$$\log 7 + x \log 2 = (x-2)\log 5$$
$$\log 7 + x \log 2 = x \log 5 - 2 \log 5$$
$$\log 7 + 2 \log 5 = x \log 5 - x \log 2$$
$$\log 7 + 2 \log 5 = x(\log 5 - \log 2)$$
$$x(\log 5 - \log 2) = \log 7 + 2 \log 5$$
$$x = \frac{\log 7 + 2 \log 5}{\log 5 - \log 2}$$
$$\doteq 5.637$$



Section 2: Activity 3 (continued)

3. a. Textbook questions 26, 27, 30, and 31 of "Practice," p. 113

26. Use the compound interest formula, $A(t) = P(1+i)^n$.

In this question, $i = 12\% = 0.12$.

Because the interest is compounded annually, $n = t$.

$\therefore A(t) = P(1.12)^t$

When the original amount doubles, $A(t) = 2P$.

$$\begin{aligned}
\therefore A(t) &= P(1.12)^t \\
2P &= P(1.12)^t \\
2 &= 1.12^t \\
\log 2 &= \log 1.12^t \\
\log 2 &= t \log 1.12 \\
t &= \frac{\log 2}{\log 1.12} \\
&\doteq 6.1
\end{aligned}$$

Because the interest is calculated annually, the time must be rounded up to 7 years. Therefore, it will take 7 years for the amount to at least double in value.

27. Use the compound interest formula, $A(t) = P(1+i)^n$.

Because the interest is calculated monthly, the rate per period is $i = \frac{12\%}{12} = 0.01$.

Because the interest is compounded monthly, there are $12t$ interest periods in t years. Therefore, $n = 12t$.

$\therefore A(t) = P(1.01)^{12t}$

When the original amount doubles, $A(t) = 2P$.

$$\therefore A(t) = P(1.01)^{12t}$$
$$2P = P(1.01)^{12t}$$
$$2 = 1.01^{12t}$$
$$\log 2 = \log 1.01^{12t}$$
$$\log 2 = 12t \log 1.01$$
$$t = \frac{\log 2}{12 \log 1.01}$$
$$\doteq 5.8$$

$0.8 \text{ years} \times 12 \text{ months/year} = 9.6 \text{ months}$

Because interest is calculated monthly, it will take 5 years 10 months for the original investment to double.

30. Let $V(t)$ represent the value of the photocopier after t years.

Since its original value was \$12 500 and it is worth 85% of its preceding year's value, the equation that models this situation is $V(t) = 12\,500(0.85)^t$.

31. Substitute $V(t) = 6250$ into the equation.

$$V(t) = 12\,500(0.85)^t$$
$$6250 = 12\,500(0.85)^t$$
$$\frac{6250}{12\,500} = 0.85^t$$
$$0.5 = 0.85^t$$
$$\log 0.5 = \log 0.85^t$$
$$\log 0.5 = t \log 0.85$$
$$t = \frac{\log 0.5}{\log 0.85}$$
$$\doteq 4.3$$

It will take approximately 4.3 years before the photocopier is worth half its original cost.

Section 2: Activity 3 (continued)

b. Textbook questions 43, 44, 47, and 53 of "Applications and Problem Solving," pp. 114 and 115

43. The general equation is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$, where $A(t)$ represents the amount present after t days, A_0 is the original amount, and H is the half-life of the substance.

$A(t) = 16.85$ mg, $A_0 = 20$ mg, and $H = 8.1$ days.

$$\therefore A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$$

$$16.85 = 20\left(\frac{1}{2}\right)^{\frac{t}{8.1}}$$

$$\frac{16.85}{20} = \left(\frac{1}{2}\right)^{\frac{t}{8.1}}$$

$$\log\frac{16.85}{20} = \frac{t}{8.1}\log\frac{1}{2}$$

$$t = \frac{8.1\log\frac{16.85}{20}}{\log\frac{1}{2}}$$

$$\doteq 2.0$$

It will take approximately 2.0 days for 20 mg of iodine-131 to decay to 16.85 mg.

44. a. $P(t) = 3200$, $P_0 = 20$, and $t = 3$ h

$$\therefore P(t) = P_0(8)^{kt}$$

$$3200 = 20(8)^{3k}$$

$$160 = 8^{3k}$$

$$\log 160 = \log 8^{3k}$$

$$\log 160 = 3k\log 8$$

$$k = \frac{\log 160}{3\log 8}$$

$$\doteq 0.81$$

The value of k is approximately 0.81.

b. $P(t) = 3P_0$ and $k = 0.81$

$$\therefore P(t) = P_0(8)^{kt}$$

$$3P_0 \doteq P_0(8)^{0.81t}$$

$$3 \doteq 8^{0.81t}$$

$$\log 3 \doteq \log 8^{0.81t}$$

$$\log 3 \doteq 0.81t\log 8$$

$$t \doteq \frac{\log 3}{0.81\log 8}$$

$$\doteq 0.65$$

It will take about 0.65 h for the bacteria population to triple.

c. $P(t) = 0.5P_0$ and $k = 0.81$

$$\therefore P(t) = P_0 (8)^{kt}$$
$$0.5P_0 = P_0 (8)^{0.81t}$$
$$0.5 = 8^{0.81t}$$
$$\log 0.5 = \log 8^{0.81t}$$
$$\log 0.5 = 0.81t \log 8$$
$$t = \frac{\log 0.5}{0.81 \log 8}$$
$$\doteq -0.41$$

There were half as many bacteria approximately 0.41 h ago.

47. a. The general equation is $A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{H}}$, where $A(t)$ represents the amount present after t years, A_0 is the original amount, and H is the half-life.

The half-life, H, of strotium-90 is 28.8 years. Therefore, the equation is $A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$.

b. $A(t) = 10\%$ of A_0
$$= 0.1A_0$$

$$\therefore A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$$
$$0.1A_0 = A_0 \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$$
$$0.1 = 0.5^{\frac{t}{28.8}}$$
$$\log 0.1 = \log 0.5^{\frac{t}{28.8}}$$
$$-1 = \frac{t}{28.8} \log 0.5$$
$$t = \frac{-28.8}{\log 0.5}$$
$$\doteq 95.7$$

$\log 0.1 = \log_{10} 10^{-1}$
$= -1$

It will take approximately 95.7 years before there is only 10% of the original amount remaining.

Section 2: Activity 3 (continued)

c. After 5 years, $t = 5$.

$$\therefore \text{ Percentage} = \frac{A(5)}{A_0} \times 100\%$$

$$= \frac{A_0 \left(\frac{1}{2}\right)^{\frac{5}{28.8}}}{A_0} \times 100\%$$

$$= \left(\frac{1}{2}\right)^{\frac{5}{28.8}} \times 100\%$$

$$\doteq 89\%$$

After 5 years, approximately 89% of strontium-90 remains.

d. $A(t) = 25\% \text{ of } A_0$

$$= \frac{25}{100} A_0$$

$$= \frac{1}{4} A_0$$

$$\therefore A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$$

$$\frac{1}{4} A_0 = A_0 \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$$

$$\frac{1}{4} = \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$$

$$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{\frac{t}{28.8}}$$

$$2 = \frac{t}{28.8}$$

$$t = 2(28.8)$$

$$= 57.6$$

It will take 57.6 years before only 25% of the strontium-90 remains.

53. Let $I(d)$ be the intensity of light at any depth, d (in metres), below the surface.

Since 4.6% is absorbed per metre, 95.4% is transmitted.

If I_0 is the intensity of light that strikes the water's surface, then the equation is $I(d) = I_0 (0.954)^d$.

Solve for d.

$$I(d) = I_0 (0.954)^d$$

$$\frac{I(d)}{I_0} = 0.954^d$$

$$\log \frac{I(d)}{I_0} = \log 0.954^d$$

$$\log \frac{I(d)}{I_0} = d \log 0.954$$

$$d = \frac{1}{\log 0.954} \times \log \frac{I(d)}{I_0}$$

$$\doteq -48.9 \log \frac{I(d)}{I_0}$$

4. a. Textbook questions 15 and 19 of "Practice," p. 113

15. The value of $\log_3 88$ is slightly larger than 4 because $3^4 = 81$.

Let $x = \log_3 88$.

$$\therefore\ 3^x = 88$$

$$\log 3^x = \log 88$$

$$x \log 3 = \log 88$$

$$x = \frac{\log 88}{\log 3}$$

$$\doteq 4.08$$

$$\therefore\ \log_3 88 \doteq 4.08$$

Section 2: Activity 3 (continued)

19. The value of $\log_7 71$ is between 2 and 3 because $7^2 = 49$ and $7^3 = 343$.

Let $x = \log_7 71$.

$$\therefore\ 7^x = 71$$
$$\log 7^x = \log 71$$
$$x \log 7 = \log 71$$
$$x = \frac{\log 71}{\log 7}$$
$$\doteq 2.19$$

$$\therefore\ \log_7 71 \doteq 2.19$$

b. Textbook question 41 of "Applications and Problem Solving," p. 114

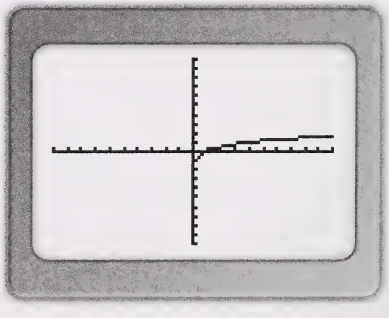
41. Rewrite the function using the change of base formula.

$$y = \log_4 x$$
$$= \frac{\log x}{\log 4}$$

$\log_b a = \dfrac{\log_c a}{\log_c b}$

To graph this function on your graphing calculator, press the following keystrokes:

Y= LOG X,T,Θ,*n*) ÷ LOG 4) GRAPH



To verify your graph using $x = 4$, press 2nd TABLE to see if the point $(4, 1)$ lies on the graph.

Remember, $\log_4 4 = 1$.

You could also use any convenient power of 4, such as 16 and 64, to verify your graph.

See if $\log_4 16 = 2$ or $\log_4 64 = 3$.

5. a. Textbook questions 33.b. and 35.b. of "Practice," p. 113

33. b.

LS	RS
$(\log_a x)(\log_x a)$	1
$=\left(\frac{\log x}{\log a}\right)\left(\frac{\log a}{\log x}\right)$	
$=1$	

LS = RS

$\therefore\ (\log_a x)(\log_x a)=1$

$\log_b a=\frac{\log_c a}{\log_c b}$

35. b.

LS	RS
$(\log_a x)(\log_b a)$	$\log_b x$
$=\left(\frac{\log x}{\log a}\right)\left(\frac{\log a}{\log b}\right)$	$=\frac{\log x}{\log b}$
$=\frac{\log x}{\log b}$	

LS = RS

$\therefore\ (\log_a x)(\log_b a)=\log_b x$

b. Textbook questions 48 and 51 of "Applications and Problem Solving," pp. 114 and 115

48.

LS	RS
$\log_a b$	$\frac{1}{\log_b a}$
$=\frac{\log b}{\log a}$	$=\frac{1}{\frac{\log a}{\log b}}$
	$=\frac{\log b}{\log a}$

LS = RS

$\therefore\ \log_a b=\frac{1}{\log_b a}$, where $a>0,\ b>0,\ a\neq 1,$ and $b\neq 1$

Section 2: Activity 3 (continued)

51.

LS	RS
$\left(\log\frac{a}{b}\right)\left(\log\frac{c}{d}\right)$	$\left(\log\frac{a}{c}\right)\left(\log\frac{b}{d}\right)+\left(\log\frac{a}{d}\right)\left(\log\frac{c}{b}\right)$
$=(\log a-\log b)(\log c-\log d)$	$=(\log a-\log c)(\log b-\log d)$
$=(\log a)(\log c)-(\log a)(\log d)$	$\quad+(\log a-\log d)(\log c-\log b)$
$\quad-(\log b)(\log c)+(\log b)(\log d)$	$=\cancel{(\log a)(\log b)}-(\log a)(\log d)-(\log c)(\log b)$
	$\quad+\cancel{(\log c)(\log d)}+(\log a)(\log c)-\cancel{(\log a)(\log b)}$
	$\quad-\cancel{(\log d)(\log c)}+(\log d)(\log b)$
	$=-(\log a)(\log d)-(\log c)(\log b)+(\log a)(\log c)$
	$\quad+(\log d)(\log b)$
	$=(\log a)(\log c)-(\log a)(\log d)-(\log b)(\log c)$
	$\quad+(\log b)(\log d)$

LS = RS

$\therefore \left(\log\frac{a}{b}\right)\left(\log\frac{c}{d}\right)=\left(\log\frac{a}{c}\right)\left(\log\frac{b}{d}\right)+\left(\log\frac{a}{d}\right)\left(\log\frac{c}{b}\right)$

6. a. Textbook questions 21 and 24 of "Practice," p. 113

21. $\log_2 x=\log_2 5+\log_2 3$

$\log_2 x=\log_2 (5\times 3)$

$\log_2 x=\log_2 15$

$x=15$

24. $\log x=1+\log 2$

$\log x-\log 2=1$

$\log\frac{x}{2}=1$

$\frac{x}{2}=10^1$ ← Write in exponential form.

$x=20$

b. Textbook questions 40.a., 40.c., 40.f., and 62.c. of "Applications and Problem Solving," pp. 114 and 115

40. a. Because $\log_b n$ exists only for $n > 0$, the restriction is $x > 2$.

Solve for x.

$$\log_2 (x-2) + \log_2 x = \log_2 3$$
$$\log_2 [(x-2)x] = \log_2 3$$
$$(x-2)x = 3$$
$$x^2 - 2x = 3$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$

$$\therefore\ x-3=0 \quad \text{or} \quad x+1=0$$
$$x=3 \qquad\qquad x=-1$$

The solution $x = -1$ must be rejected because of the restriction $x > 2$. Therefore, the solution is $x = 3$.

c. Because $\log_b n$ exists only for $n > 0$, the restriction is $x > 3$.

Solve for x.

$$\log_5 (3x+1) + \log_5 (x-3) = 3$$
$$\log_5 [(3x+1)(x-3)] = 3$$
$$(3x+1)(x-3) = 5^3$$
$$3x^2 - 9x + x - 3 = 125$$
$$3x^2 - 8x - 128 = 0$$
$$(3x+16)(x-8) = 0$$

$$\therefore\ 3x+16=0 \quad \text{or} \quad x-8=0$$
$$3x=-16 \qquad\qquad x=8$$
$$x=-\frac{16}{3}$$

The solution $x = -\frac{16}{3}$ must be rejected because of the restriction $x > 3$. Therefore, the solution is $x = 8$.

Section 2: Activity 3 (continued)

f. Because $\log_b n$ exists only for $n > 0$, the restriction is $x > 2$.

Solve for x.

$$\log(2x+1) = 1 + \log(x-2)$$
$$\log(2x+1) - \log(x-2) = 1$$
$$\log\frac{2x+1}{x-2} = 1$$
$$\frac{2x+1}{x-2} = 10^1$$
$$2x+1 = 10(x-2)$$
$$2x+1 = 10x-20$$
$$-8x = -21$$
$$x = \frac{21}{8}$$

Since $\frac{21}{8} > 2$, the solution is $x = \frac{21}{8}$.

62. c. Because $\log_b n$ exists only for $n > 0$, the restriction is $x > 3$.

Solve for x.

$$\log_{36}(x-2) + \log_{36}(x+1) + \log_{36}(x-3) = 4^{-\frac{1}{2}}$$
$$\log_{36}[(x-2)(x+1)(x-3)] = \frac{1}{4^{\frac{1}{2}}}$$
$$\log_{36}[(x-2)(x+1)(x-3)] = \frac{1}{2}$$
$$(x-2)(x+1)(x-3) = 36^{\frac{1}{2}}$$
$$(x-2)(x^2-2x-3) = \sqrt{36}$$
$$x^3 - 2x^2 - 3x - 2x^2 + 4x + 6 = 6$$
$$x^3 - 4x^2 + x + 6 = 6$$
$$x^3 - 4x^2 + x = 0$$
$$x(x^2 - 4x + 1) = 0$$

$\therefore\ x = 0 \quad$ or $\quad x^2 - 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

Note:
$a = 1$
$b = -4$
$c = 1$

Since $x > 3$, the only admissible root is $2 + \sqrt{3}$. Therefore, the solution is $x = 2 + \sqrt{3}$.

7. a. Textbook questions 1 to 6 of Investigation 1, "Comparing the Intensity of Sounds," pp. 124 and 125

1. Substitute 45 for dB.

$$\text{dB} = 10 \log \frac{I}{I_0}$$

$$45 = 10 \log \frac{I}{I_0}$$

$$4.5 = \log \frac{I}{I_0}$$

$$\frac{I}{I_0} = 10^{4.5}$$

$$I = 10^{4.5} I_0$$

If the decibel level of a sound is 45 dB, then the intensity is $10^{4.5} I_0$.

Section 2: Activity 3 (continued)

2. $$\frac{I_{\text{rock concert}}}{I_{\text{loud stereo}}} = \frac{10^{12} I_0}{10^{8} I_0}$$
$$= 10^{4}$$
$$= 10\,000$$

A rock concert is 10 000 times louder than a loud stereo.

3. $$\frac{I_{\text{boom box}}}{I_{\text{traffic noise}}} = \frac{10^{8.5} I_0}{10^{8.0} I_0}$$
$$= 10^{0.5}$$
$$\doteq 3$$

The intensity of an 85-dB sound is $10^{8.5} I_0$. The intensity of an 80-dB sound is $10^{8.0} I_0$.

The intensity of a boom box is about 3 times greater than the intensity of maximum allowable traffic noise.

4. Substitute 1 for dB.

$$\text{dB} = 10 \log \frac{I}{I_0}$$
$$1 = 10 \log \frac{I}{I_0}$$
$$0.1 = \log \frac{I}{I_0}$$
$$\frac{I}{I_0} = 10^{0.1}$$
$$I = 10^{0.1} I_0$$
$$\doteq 1.3 I_0$$

The intensity of the difference in sound that the human ear can detect is approximately 1.3 times louder than the intensity of a barely audible sound, I_0.

5. Compare the intensity of the 100 dB and 95 dB sounds.

$$\begin{aligned}\frac{I_{100\text{ dB}}}{I_{95\text{ dB}}} &= \frac{10^{10} I_0}{10^{9.5} I_0} \\ &= 10^{0.5} \\ &\doteq 3\end{aligned}$$

A 100-dB sound is about 3 times louder than a 95-dB sound. Therefore, management's reduction of the noise level from 100 dB to 95 dB is a significant reduction.

6. The use of decibels, instead of Bels, emphasizes how the loudness or intensities of two different sounds compare. For every 10 dB, there is a tenfold increase in the loudness, or intensity, of the sound. For example, a 30-dB sound is 10 times louder than a 20-dB sound.

b. Textbook questions 1 and 2 of Investigation 2, "The Effect of Combining Sounds," p. 125

1. Let $10^x I_0$ represent the intensity of the noise produced by the two planes taking off at the same time.

$$\begin{aligned}\therefore\ 10^x I_0 &= 2\times 10^{11} I_0 \\ 10^x &= 2\times 10^{11} \\ x &= \log\left(2\times 10^{11}\right) \\ &= \log 2 + \log 10^{11} \\ &= \log 2 + 11 \\ &\doteq 11.3\end{aligned}$$

Therefore, the decibel level of the two planes is approximately 113 dB.

2. Let $10^x I_0$ represent the intensity of sound of the three race cars.

$$\begin{aligned}\therefore\ 10^x I_0 &= 3\times 10^{8.5} I_0 \\ 10^x &= 3\times 10^{8.5} \\ x &= \log\left(3\times 10^{8.5}\right) \\ &= \log 3 + \log 10^{8.5} \\ &= \log 3 + 8.5 \\ &\doteq 9.0\end{aligned}$$

Therefore, the decibel level of three race cars accelerating at the same time is approximately 90 dB.

Section 2: Follow-up Activities

Extra Help

1. Textbook questions 16, 21, 22, 27, 33, 37, and 41 of "Review," pp. 126 and 127

16. Method 1: Using an Exponential Equation

$$x = \log_3 243$$
$$3^x = 243$$
$$3^x = 3^5$$
$$x = 5$$

Method 2: Using Logarithms

$$x = \log_3 243$$
$$= \log_3 3^5$$
$$= 5$$

21. $\log_8 x = 2$

$$x = 8^2 \quad \text{Write in exponential form.}$$
$$= 64$$

22. $\log_x 81 = 4$

$$x^4 = 81 \quad \text{Write in exponential form.}$$
$$x = \sqrt[4]{81}$$
$$= 3 \quad \text{Remember, } x > 0.$$

27.
$$3\log A + 2\log B - \left[\log\sqrt{A} - \log(2B)\right] = \log A^3 + \log B^2 - \log\sqrt{A} + \log(2B)$$
$$= \left[\log A^3 + \log B^2 + \log(2B)\right] - \log\sqrt{A}$$
$$= \log\left(A^3 \times B^2 \times 2B\right) - \log\sqrt{A}$$
$$= \log\left(2A^3B^3\right) - \log\sqrt{A}$$
$$= \log\frac{2A^3B^3}{\sqrt{A}}$$
$$= \log\frac{2A^3B^3}{A^{\frac{1}{2}}}$$
$$= \log\left(2A^{\frac{5}{2}}B^3\right)$$

33. $5^x = 241$

$\log 5^x = \log 241$ ← Take the logarithm of both sides.

$x \log 5 = \log 241$ ← Logarithm of a Power Law

$$x = \frac{\log 241}{\log 5}$$

$$\doteq 3.41$$

37. $x = \log_2 57$

$2^x = 57$ ← Write in exponential form.

$\log 2^x = \log 57$ ← Take the logarithm of both sides.

$x \log 2 = \log 57$ ← Logarithm of a Power Law

$$x = \frac{\log 57}{\log 2}$$

$$\doteq 5.83$$

41. $\log_4 (x+3) - 2 = \log_4 (x-4)$ ← Notice that the restriction is $x > 4$.

$\log_4 (x+3) - \log_4 (x-4) = 2$

$$\log_4 \frac{x+3}{x-4} = 2$$ ← Logarithm of a Quotient Law

$$\frac{x+3}{x-4} = 4^2$$ ← Write in exponential form.

$$\frac{x+3}{x-4} = 16$$

$$x+3 = 16(x-4)$$

$$x+3 = 16x - 64$$

$$67 = 15x$$

$$x = \frac{67}{15}$$

Since $\frac{67}{15} > 4$, the solution is $x = \frac{67}{15}$.

Extra Help (continued)

2. Textbook questions 6.a., 8.a., 9.b., and 10.b. of "Chapter Check," p. 128

6. a. $\log_3 16 - 2\log_3 32 = \log_3 4^2 - \log_3 32^2$

$= \log_3 4^2 - \log_3 1024$

$= \log_3 4^2 - \log_3 4^5$

$= 2\log_3 4 - 5\log_3 4$

$= 2x - 5x$

$= -3x$

8. a. $\log y = \log(1-3x)$

First, determine the domain.

$\log(1-3x)$ is defined if

$1-3x > 0$

$-3x > -1$

$x < \frac{1}{3}$

Now, solve for y.

$\log y = \log(1-3x)$

$y = 1-3x$, where $x < \frac{1}{3}$

9. b. $5^{2x+1} = 498$

$\log 5^{2x-1} = \log 498$

$(2x+1)\log 5 = \log 498$

$2x\log 5 + \log 5 = \log 498$

$2x\log 5 = \log 498 - \log 5$

$x = \dfrac{\log 498 - \log 5}{2\log 5}$

$\doteq 1.43$

10. b. $\log_2(33x-9)-\log_2(2x+3)=4$

First, find the restriction.

$$\begin{array}{lll} 33x-9>0 & \text{and} & 2x+3>0 \\ 33x>9 & & 2x>-3 \\ x>\frac{9}{33} & & x>-\frac{3}{2} \\ x>\frac{3}{11} & & \end{array}$$

The restriction is $x>\frac{3}{11}$.

Next, combine logarithms and solve for x.

$$\begin{aligned} \log_2(33x-9)-\log_2(2x+3)&=4 \\ \log_2\frac{33x-9}{2x+3}&=4 \\ \frac{33x-9}{2x+3}&=2^4 \quad \text{Write in exponential form.} \\ \frac{33x-9}{2x+3}&=16 \\ 33x-9&=16(2x+3) \\ 33x-9&=32x+48 \\ x&=57 \end{aligned}$$

Since $57>\frac{3}{11}$, the solution is $x=57$.

Enrichment

1. a. Textbook question "Explore: Use a Table," p. 116

Principal	Number of Compounding Periods	Rate of Interest Per Period	Amount in Dollars
\$1	1	1.00	$1(1+1.00)^1 = 2$
\$1	2	0.50	$1(1+0.50)^2 = 2.25$
\$1	4	0.25	$1(1+0.25)^4 = 2.44$
\$1	12	$\frac{1}{12}$	$1\left(1+\frac{1}{12}\right)^{12} = 2.61$
\$1	100	0.01	$1(1+0.01)^{100} = 2.70$
\$1	1000	0.001	$1(1+0.001)^{1000} = 2.72$
\$1	10 000	0.0001	$1(1+0.0001)^{10\,000} = 2.72$
\$1	100 000	0.000 01	$1(1+0.000\,01)^{100\,000} = 2.72$

b. Textbook questions 1 to 5 of "Inquire," p. 116

1. As the frequency of compounding increases, the investment approaches \$2.72.

2. You would expect $500 \times \$2.72 = \$1360$.

3. If the compound amount was \$2125.70 after 1 year, the original investment was $\frac{\$2125.70}{2.72} = \$781.51$.

4. $A = 1\left(1+\frac{1}{n}\right)^n$ or $A = \left(1+\frac{1}{n}\right)^n$

5. The intensity of sound decreases continuously as you move away from its source, whereas minimum wages increase by multiples of cents. Therefore, the intensity of sound is an example of continuous exponential growth and increases in minimum wage is not.

2. a. Textbook questions 1, 3, 7, 13, 15, and 25 of "Practice," p. 119

1. $\ln 2 \doteq 0.693$

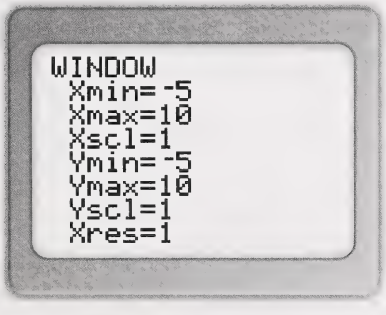
3. $\ln 0.3 = -1.204$

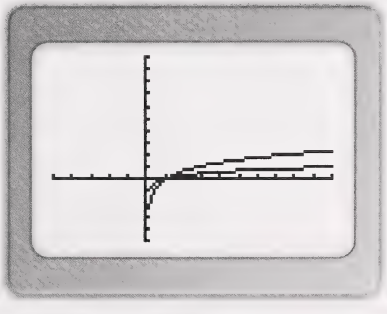
7. $e^{0.6} \doteq 1.822$

13.
$$1500 = 5e^{0.045x}$$
$$\ln 1500 = \ln\left(5e^{0.045x}\right)$$
$$\ln 1500 = \ln 5 + \ell n\, e^{0.045x}$$
$$\ln 1500 = \ln 5 + 0.045x$$
$$\ln 1500 - \ln 5 = 0.045x$$
$$x = \frac{\ln 1500 - \ln 5}{0.045}$$
$$\doteq 126.751$$

15.
$$65 = e^{7n}$$
$$\ln 65 = \ell n\, e^{7n}$$
$$\ln 65 = 7n$$
$$n = \frac{\ln 65}{7}$$
$$\doteq 0.596$$

25. The graph of $y = \log x$ lies below the graph of $y = \ln x$ for $x > 1$.





The graphs cross at $(1, 0)$ because $\log 1 = 0$ and $\ln 1 = 0$.

Enrichment (continued)

b. Textbook questions 26, 27, and 32 of "Applications and Problem Solving," pp. 119 and 120

26. a. The year 2000 is 3 years after 1997. Therefore, $t = 3$.

$$\begin{aligned}\therefore\ P &= 795e^{0.0306t} \\ &= 795e^{0.0306(3)} \\ &= 795e^{0.0918} \\ &\doteq 871\end{aligned}$$

The population of Calgary in the year 2000 is approximately 871 000.

b. Using $P = 795e^{0.0306t}$, for the population to exceed one million, $P > 1000$.

$$\begin{aligned}\therefore\ 795e^{0.0306t} &> 1000 \\ \ln\left(795e^{0.0306t}\right) &> \ln 1000 \\ \ln 795 + \ln e^{0.0306t} &> \ln 1000 \\ \ln 795 + 0.0306t &> \ln 1000 \\ 0.0306t &> \ln 1000 - \ln 795 \\ t &> \frac{\ln 1000 - \ln 795}{0.0306} \\ &> 7.50\end{aligned}$$

The population of Calgary should exceed one million approximately 7.5 years after 1997, or sometime in the year 2005.

c. Answers may vary. Calgary's population may deviate from the predicted population if the yearly percentage increase in population varies from 3.06%.

27. a. When first poured, $t = 0$.

$$\begin{aligned}\therefore\ T &= 95e^{-0.05t} \\ &= 95e^{-0.05(0)} \\ &= 95e^{0} \\ &= 95(1) \\ &= 95\end{aligned}$$

When first poured, the temperature of the coffee was 95°C.

b. Ten minutes later, $t = 10$.

$$\begin{aligned}\therefore T &= 95e^{-0.05t} \\ &= 95e^{-0.05(10)} \\ &= 95e^{-0.50} \\ &\doteq 58\end{aligned}$$

Ten minutes later, the temperature of the coffee is 58°C.

32. a. Use $T - T_0 = ae^{-kt}$

At 1:30 A.M., $T = 32$, $T_0 = 10$, and $t = 0$.

$$\begin{aligned}\therefore T - T_0 &= ae^{-kt} \\ 32 - 10 &= ae^{-k(0)} \\ 22 &= ae^0 \\ 22 &= a(1) \\ a &= 22\end{aligned}$$

At 3:00 A.M., $T = 28$, $T_0 = 10$, and $t = 90$.

$$\begin{aligned}\therefore T - T_0 &= ae^{-kt} \\ 28 - 10 &= 22e^{-k(90)} \\ 18 &= 22e^{-90k} \\ \ln 18 &= \ln\left(22e^{-90k}\right) \\ \ln 18 &= \ln 22 + \ln e^{-90k} \\ \ln 18 &= \ln 22 - 90k \\ 90k &= \ln 22 - \ln 18 \\ k &= \frac{\ln 22 - \ln 18}{90} \\ &\doteq 0.002\,23\end{aligned}$$

Therefore $a = 22$ and $k \doteq 0.002\,23$.

Enrichment (continued)

b. Use $T - T_0 = ae^{-kt}$

At the time of death, $T = 37$, $T_0 = 10$, $a = 22$, and $k \doteq 0.002\,23$.

$$\therefore\ T - T_0 = ae^{-kt}$$
$$37 - 10 \doteq 22e^{-0.002\,23t}$$
$$27 \doteq 22e^{-0.002\,23t}$$
$$\ln 27 \doteq \ln 22 - 0.002\,23t$$
$$0.002\,23t \doteq \ln 22 - \ln 27$$
$$t = \frac{\ln 22 - \ln 27}{0.002\,23}$$
$$\doteq -90$$

The time of death occurred approximately 90 minutes prior to its discovery at 1:30 A.M.; therefore, the approximate time of death was midnight.

The assumption made is that the basement temperature remained steady at 10°C.

c.
$$T - T_0 = ae^{-kt}$$
$$\ln(T - T_0) = \ln(ae^{-kt})$$
$$\ln(T - T_0) = \ln a + \ln e^{-kt}$$
$$\ln(T - T_0) = \ln a - kt$$
$$kt = \ln a - \ln(T - T_0)$$
$$t = \frac{\ln a - \ln(T - T_0)}{k}$$
$$= -\frac{\ln(T - T_0) - \ln a}{k}$$
$$= -\frac{1}{k}\ln\frac{T - T_0}{a}$$

Credits

Some clip art graphics are commercially owned.

Welcome Page

Image Club/Studio Gear/EyeWire, Inc.

Page

2–3 **collage:** PhotoDisc, Inc.
6 PhotoDisc, Inc.
8 Image Club/Studio Gear/EyeWire, Inc.
19 Image Club/Studio Gear/EyeWire, Inc.
21 PhotoDisc, Inc.
33 Image Club/Studio Gear/EyeWire, Inc.
34 Image Club/Studio Gear/EyeWire, Inc.
35 Image Club/Studio Gear/EyeWire, Inc.
36 PhotoDisc, Inc.
37 Image Club/Studio Gear/EyeWire, Inc.
39 Image Club/Studio Gear/ EyeWire, Inc.
40 **bottom:** PhotoDisc, Inc.
41 Image Club/Studio Gear/EyeWire, Inc.
42 **bottom:** Corel Corporation
43 Image Club/Studio Gear/EyeWire, Inc.
44 EyeWire, Inc.
45 **top:** PhotoDisc, Inc.
bottom: EyeWire, Inc.
49 **top:** Image Club/Studio Gear/EyeWire, Inc.
bottom: PhotoDisc, Inc.
52 Corel Corporation
54 EyeWire, Inc.
55 **top:** Image Club/Studio Gear/EyeWire, Inc.
bottom: Corel Corporation
56 Image Club/Studio Gear/EyeWire, Inc.
57 Image Club/Studio Gear/EyeWire, Inc.

Pure Mathematics 30
Student Module Booklet
Module 2

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